A Path to Inquiry-Based Learning in Geometry Courses in U.S. Secondary Schools

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Abstract

This thesis outlines an approach for teachers to improve student interest in and understanding of mathematics through inquiry-based learning (IBL) in high school geometry courses. A review of four methods of IBL and its effects with students finds compelling evidence for teachers to transition from direct instruction to IBL. The challenges of implementation of IBL are examined in interviews of working and retired mathematics teachers. The thesis offers three example inquiry lessons with guidelines to create more. The example lessons were taught to secondary students, and a survey of those students indicates that IBL lessons can be successfully implemented with minimal disruption. We conclude that inquiry-based learning offers advantages to students, and therefore teachers should take steps to adopt IBL in their classrooms. The thesis provides a pathway for individual teachers to begin to adopt IBL, but further work can be done to explore how IBL might be effectively included as part of standardized curricula.
Author’s Biographical Sketch

Joel Patterson attended the Arkansas Governor’s School in 1990 and graduated from Conway High School, in Conway, Arkansas, in 1991. He earned a Bachelor of Arts degree in Physics from Rice University in 1995 and a teacher certification from University of Houston in 1998. He has taught mathematics in Houston, Texas, Seattle, Washington, and Cambridge, Massachusetts.
Dedication

To my first teachers, my parents,

Raymond and Shirley Patterson,

who always encouraged me to ask questions.
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Chapter 1 Introduction

Mathematics is valued across cultures for its power to model and solve complicated problems and for its own simple beauty. However, mathematics education is itself not simple. Educators may aim to build student confidence, communication skills, fluency or appreciation of mathematics, but ultimately a principal goal must be understanding and mathematical thinking. When student performances of understanding are analyzed, visible gaps can be seen among groups of students. Some frustrating performances of student understanding have been observed wherein students memorize rules and then apply the rules in a manner which is not sensible. In the article “How Old Is The Shepherd?” one mathematical question posed to young children illustrates this failure of thinking. "There are 125 sheep and 5 dogs in a flock. How old is the shepherd? Researchers report that three out of four schoolchildren will produce a numerical answer to this problem” (Merseth, 1993). Obviously, there is no reasonable way to use the numerical data provided to answer this question, but 75% of students do exactly this. Merseth blames this error on (among other causes) “the typical curriculum in use in our classrooms,” (Merseth, 1993) which emphasizes solving problems by repeating rules without thinking through questions on a sense-making level. Students are trained to accept questions uncritically and, as a result, produce less sensible or even nonsensical answers to mathematical questions. This approach is the very antithesis of mathematical thinking, and mathematics teachers of all levels have their own similar personal experiences of some student work that showed at most rote application of rules and little
sense-making. While there may be additional causes for this illogical application of rules to math questions, when Merseth pinpointed the typical curriculum, she gave mathematics teachers a great starting point for how to improve math education in their classrooms.

Even when the curriculum is updated to address concerns of mathematical thinking and understanding specifically, mathematics educators may be hampered by the canonical resources at their disposal. According to the Common Core State Standards, if a student is going to engage in a mathematical modeling task, they should work through five subtasks (CCSSI 2010, p. 72):

1. Identifying essential variables in a situation
2. Formulating models from those variables
3. Performing operations using those models
4. Interpreting the results of those operations
5. Validating the conclusions of those results

Each of these subtasks, except the third, emphasizes mathematical thinking and sense-making. But a recent survey (Meyer, 2015) of the modeling problems in one publisher’s Algebra I and Geometry textbooks reveals that this refocusing of the curriculum is not reflected in the student resources.

Meyer found that of 83 problems labeled “modeling” in the book, merely seven expected students to do the identifying, twenty expected students to formulate the model, and only four expected students to validate their final result. Practicing those subtasks (or “actions,” in CCSS parlance) would give students much more experience in making sense of questions, choosing the right relationships among variables, and double-
checking their answers. If sense-making is to become a habit of mind, then it must be practiced regularly. No habit of mind is going to be inculcated by students who practice it merely four, seven, or twenty times over the span of two math courses. Meyer concludes that teachers must make up the gap left by the resources readily at hand. They must create lessons where students identify variables, formulate models, and judge the sensibility of results.

Misguided curricula and teaching resources that miss opportunities to realize even well-focused curriculum goals do not just fail to develop student thinking and understanding. A common criticism of math education is that it bores students or, even if students have fun activities, the activities fail to engage the students in the mathematics itself. In his essay, “A Mathematician’s Lament,” Paul Lockhart (2002) makes perhaps the most well-known critique of typical math curricula, which he likens to the music education nightmare of forcing students to learn (from a non-musician) formal music theory and sheet music notation without ever having the chance to sing or compose a piece or appreciate a recording. He describes how this type of math education persists:

The cultural problem is a self-perpetuating monster: students learn about math from their teachers, and teachers learn about it from their teachers, so this lack of understanding and appreciation for mathematics in our culture replicates itself indefinitely. Worse, the perpetuation of this “pseudo-mathematics,” this emphasis on the accurate yet mindless manipulation of symbols, creates its own culture and its own set of values. Those who have become adept at it derive a great deal of self-esteem from their success. The last thing they want to hear is that math is really about raw creativity and aesthetic sensitivity. Many a graduate student has come to grief when they discover, after a decade of being told they were “good at math,” that in fact they have no real mathematical talent
and are just very good at following directions. Math is not about following directions, it’s about making new directions. (Lockhart, 2002, p. 6)

The term ‘pseudo-mathematics’ is a harsh one though not misused: when students merely memorize steps to solve equations or write geometric proofs, they can be numb to the actual logical reasons for the steps; they can overlook the opportunities to tighten up solutions or proofs with elegant moves; moreover, they are doomed to miss out on the thrill of discovery, the pleasure of the creative process and the essence of mathematics.

According to Lockhart, mathematics should be seen as a creative, almost artistic pursuit, and good instruction should develop creative mathematical thinking. But typical curricula slow the creative, alert functions of the mind with repetition. It is no wonder, then, that the resulting boredom and attrition from mathematical courses of study leave few students able to appreciate mathematics, let alone prepared to study higher mathematics or take their robust experience in mathematical thinking into the workforce. The 1996 National Science Foundation pamphlet *Shaping the Future*, identifies similar root causes of this attraction from mathematics and its related fields.

Too many students leave SME&T [Science, Math, Engineering & Technology] courses because they find them dull and unwelcoming. Too many new teachers enter school systems underprepared, without really understanding what science and mathematics are, and lacking the excitement of discovery and the confidence and ability to help children *engage* SME&T knowledge. Too many graduates go out into the workforce ill-prepared to solve real problems in a cooperative way, lacking the skills and motivation to continue learning. (NSF, 1996, p. iii)
Lockhart may disagree with the NSF report about the aim of mathematics education, but whether the aim is to prepare students to solve “real-world problems” in the workforce, or to engage them in authentic mathematics for its own sake, the two agree on a solution. They agree that authentic scientific and mathematical discovery are inherently interesting, engaging, and joyful, and they both wish for education to convey the true nature of the fields. The executive summary of the NSF report proposed a desired outcome with a favored approach to how to build curiosity and collaboration.

The goal … is that all students have access to supportive, excellent undergraduate education in science, mathematics, engineering, and technology, and all student learn these subjects by direct experience with the methods and processes of inquiry. (NSF, 1996)

Merseth’s example makes a case for more mathematical thinking instead of rote learning, and as the NSF proposed, this thesis will explore inquiry-based learning in the form of four methods for math classrooms in Chapter 2, along with a look at the literature of research supporting IBL. While the inclusion of inquiry is desirable, a transition to IBL will face certain barriers to entry, as we will see in Chapter 3. A principal barrier to entry is lack of good teaching resources, as Meyer’s argument suggests, and in Chapter 5, we will discuss how to select, improve, and create these better resources to take small steps of incorporating IBL into existing curriculum. Lockhart and the NSF make the case for rich and direct experience with inquiry into authentic mathematics. Therefore in Chapter 4, the thesis will present sample lessons that can be incorporated into a secondary school geometry course.
Thesis Questions

Could secondary school teachers in the U.S. improve their students’ thinking in geometry courses by transitioning away from direct instruction to more inquiry? Does the history of inquiry-based learning offer evidence of improved thinking? How could such transitions be made? This thesis will offer a path secondary school teachers may use to transition their geometry classes away from the prevalent style of the teacher lecturing and the students mimicking examples to inquiry-based learning. First, the thesis will examine the history of the literature on IBL in math including some of the types of IBL curriculum that exist and the growing body of research into the practices and benefits of IBL. Because the benefits of IBL are compelling, the author interviewed working and retired mathematics educators to learn about their experiences in IBL and their perceptions of barriers to entry when teachers choose to teach with IBL. The advantages of IBL led the author to create example lessons and practical guidelines for implementing a transition away from lectures focused on the body of mathematical knowledge to more inquiry in secondary school geometry classrooms. This path will allow teachers to exercise their own autonomy to move toward more inquiry: first, adopting inquiry from outside the usual curriculum, then creating lessons of inquiry-based learning on their own, thereby addressing the concern of traditional lectures boring students out of STEM.
Chapter 2  Four Methods of IBL

In the introduction, we considered a concern about mathematics education, and we now look to the history of the literature of IBL for insights on improving education. Indeed, the body of mathematical knowledge has been and will be grown through humans thinking through a mathematical process involving inquiry. Therefore, to be educated is both to learn the body of knowledge and to learn how to inquire into mathematics (Lockhart, 1992.) However, when the history of mathematical education is considered, there is a perceived de-emphasis of inquiry by educational systems as well as certain deficiencies in the mathematical performance of students. In the literature of mathematical education time and again, one may observe voices of mathematicians and educators calling for more inquiry-based learning (IBL) in order to improve education. A review of the literature will explore the practices and benefits of IBL. This thesis will consider methods of teaching involving inquiry as an approach to improving student thinking and understanding of mathematics.

Inquiry is the most natural form of learning. Any human mind can observe phenomena and ask, “What is the real nature of this?” One can find many expressions of the value of inquiry over time and across cultures, from the Socratic method to George Poly’a’s desire for more high school mathematics teachers to teach their students the know-how of problem-solving (Polya, 1962). Al-Hasan ibn al-Haytham (b. 965 AD) elaborated on how inquiry is essential to understanding the truth:
Therefore, the seeker after the truth is not one who studies the writings of the ancients and, following his natural disposition, puts his trust in them, but rather the one who suspects his faith in them and questions what he gathers from them, the one who submits to argument and demonstration, and not to the sayings of a human being whose nature is fraught with all kinds of imperfection and deficiency. Thus the duty of the man who investigates the writings of scientists, if learning the truth is his goal, is to make himself an enemy of all that he reads, and, applying his mind to the core and margins of its content, attack it from every side. He should also suspect himself as he performs his critical examination of it, so that he may avoid falling into either prejudice or leniency (Sabra, 2003.)

But as learning has become institutionalized in education, direct instruction has come to dominate curriculum and may not offer students the opportunity to perform critical examinations of ideas. Therefore, let us look at how the natural learning of inquiry can exist in formal curriculum by focusing on four methods of IBL from secondary schools and universities: the Moore Method, the Modified Moore Method, Harkness teaching, and the Interactive Mathematics Program.

The Moore Method

The Moore Method may be summarized by a pithy maxim of its creator R.L. Moore: “That student is taught best who is told the least” (Whyburn, 1970, p. 354). The aim is to maximize students’ autonomous thought and to minimize exposition by the teacher. Moore described how he implemented such a principle in his introductory Calculus course.
I do not, at the outset, give my calculus class either a definition of
a function as a collection of ordered pairs or a definition [of a
limit] in terms of epsilon and delta of the statement that as p
approaches a, f(p) approaches b. I proceed on the supposition that
since some or all of them have rough rudimentary ideas concerning
these notions, and I prefer to try to refine and develop these ideas
of theirs as we go along instead of beginning with definitions
which, seen too soon and too abruptly, may tend to inhibit the
development of their spatial intuition… (Moore in Parker, 2005, p.
261)

Moore understood that mathematical terms, as professors understand them in the
present, have detailed components which are necessary due to some problems from a
historical context, but students new to Calculus will be unaware of those problems from
historical context, and therefore students perceive the detailed components as arbitrary if
not confusing. The epsilon-delta definition of a limit in Calculus is a perfect example of a
detail new students might be confused by while experienced mathematicians would
appreciate the necessity of rigor in such a definition. Moore’s choice of the words “refine
and develop” is telling. Lecturing minimally, he posed questions and problems in a
careful order then gave students the time and opportunity to present their thinking. He put
his students through similar experiences of the process that math researchers on the
frontiers of the field must repeatedly do: explore rudimentary ideas then refine and
develop them into rigorous terms. As one student presented a proof, the others would
carefully check for mistakes and point them out. Moore summarized even more clearly
what his method was not.

If someone should say, “Don’t you want your students to know
what a function is?” I think I might be inclined to ask, “Do you
know what a function is?” If so, who told you and why did you
believe him? Because you thought it was the proper thing to do? And I might be inclined to ask, “Do you know what a set is and will you define the meaning of: the and that and if?” (p. 261)

Moore valued building one’s own understanding in contrast to accepting received wisdom and relying on the authority of others to know the truth. By its skepticism of received wisdom, the Moore method evoked al-Haytham’s principle of attacking ideas from all sides to seek the truth. Thus the Moore Method developed autonomous, rigorous thinking in many of his students.

His biographer John Parker (2005) makes the case that the Moore Method was essential to the impressive number of students of Moore who went on to great recognition in the field of mathematics, pointing out that among his students,

[T]hree of them followed him as president of the American Mathematical Society, three others became vice-president, and another served as secretary of the AMS for many years. Five served as president of the Mathematical Association of America and three, like Moore, became members of the National Academy of Sciences while most of the rest became highly respected and well published researchers and teachers in leading American universities. (pp. viii-ix)

Obviously, his students are impressive in their accomplishments, and many of them spoke of Moore’s classes as crucial for their education. The Moore’s method would focus on building mental capabilities (not just knowledge) in the course, be it Calculus or Point Set Topology. From John Green, a Moore PhD student who went on to earn a second PhD in Statistics:

One of the keys to the success of the Moore Method of teaching is, I think, going from strength to strength. That is we develop an
extraordinary depth of understanding in a relatively small area of mathematics. The strength this gave us can later be used as a stepping-stone to building strength in other areas. Many of Dr. Moore’s most successful mathematics student went on to make significant contributions in subjects in which they had no formal training. This is the hallmark of a great education… (Parker, 2005, p. 273)

This testimony of a student of Moore speaks to “strength” as something clearly different from knowledge of content. Another student of R.L. Moore, Raymond L. Wilder had the opinion that Moore aimed to boost the “research ability” with his Method. Wilder quoted Moore as saying, “What does information amount to compared to power?” (Parker, 2005, p. 132) The implication of this rhetorical question is that while information (or knowledge) of mathematics may be impressive, it is not nearly as impressive as the power to research new ideas and push the boundaries of mathematical knowledge further. Mary Ellen Rudin, a student of Moore, also spoke of his method as a way to develop talent for mathematical research.

The student isn’t reading a book every day. He isn’t doing the sort of homework that one does in the usual course. His work is to produce mathematics every day and by producing mathematics every day he will form a habit. When he gets a Ph.D. he will continue to ‘produce mathematics everyday.’(Whyburn, 1970)

Rudin observes that the Moore Method required different behavior from students as compared to methods involving textbooks and homework, and posits that this alternate behavior of ‘producing mathematics’ makes one ready to be a mathematical researcher. And by “ready” it is clear Rudin means the student has the “strength” or “power” as Moore termed it. Early in this thesis, a Paul Lockhart quote described graduate students
in mathematics becoming frustrated by their inability to deal with new mathematics, and
Lockhart claims, “Math is not about following directions, it’s about making new
directions.” (Lockhart, 2002) If demonstrating knowledge of content (by a student
reciting a formal definition of a function given by teacher) could be “following
directions” then R.L. Moore’s “strength” is the ability to make new directions, as
Lockhart terms it. The nurturing of “strength” or a creative approach is a goal of
mathematics education that the perspectives of Moore and Lockhart share.

How did Moore’s classrooms work? R.L. Moore gave students definitions and
axioms, then asked them to prove theorems, presenting their work on the board. He told
students that if they wanted to leave the room during a presentation that would be
understood as the student wanting to preserve their own chance to derive a proof
individually. To allow students to struggle productively with ideas, he let some theorems
go unproven for a week. He forbade collaboration or research into books or consultation
with experienced students. If he suspected a student broke those rules, he might remove
the student from the class. While Moore’s interpersonal relations with some students may
have been problematic, his method offered opportunity for students to engage in the
mathematical process of inquiring, conjecturing, criticizing, and refining ideas. His
students did deduce the theorems on their own, without being given much more than
axioms and definitions, thus developing their own power and finding new directions. The
Moore Method inspired further inquiry-based learning.
The Modified Moore Method

Professors of mathematics who look to develop “strength” often use a technique called the Modified Moore Method (MMM), which contains practices that expand the experience to a greater number of students. Donald Chalice, who describes MMM as “Socratic,” wrote one template for professors to adopt them MMM (Chalice, 1995). Chalice kept the ban on early collaboration and reading books and espoused the practice of keeping notes of the proofs available in the library (or online) to allow students to drop their pencils and fully engage in thought and discussion about the proofs on the board. In addition to proofs, MMM required students to work on easier “exercise” questions to familiarize students with the nature and implications of definitions and axioms as a way to ramp up to the challenge of proofs. Chalice valued students making mistakes at the board as a way to see better the extent or boundaries of a concept. Also Chalice insists on the professor providing office hours (as much as equal to the number of class hours) for students to consult with the professor. Chalice considered MMM successful, and not just with students he considered “excellent” but also with the ones he considered “average.” As with Moore’s students, Chalice heard stories from alumni who told him that the MMM course helped them in non-academy careers “because they had to learn to express their own ideas convincingly and forcefully to a large group” (Chalice 1995). This expression of ideas is not something that students will experience as much in a lecture course, thus we have further reason to see an advantage of IBL, in the form of the Moore Method and MMM, over lecture.
While the Moore Method and the Modified Moore Method provide alternatives to direct instruction at the undergraduate and graduate level, there are some named methods available for the secondary school level. This thesis will examine two such methods: first, Harkness Teaching, then the Interactive Mathematics Project.

Harkness Teaching

Harkness Teaching was developed at Phillips Exeter Academy in Exeter, New Hampshire, utilizing student discussion and presentation with minimal teacher exposition. The name of the method comes from a donor, Edward Harkness, who provided furniture with a certain educational intent. Instead of desks in rows, chairs are arranged about round tables for better participation in discussions. As with MMM, the teacher poses questions and lets students do the work of thinking, but discussion and collaboration are encouraged in the early stage, while MMM disallows it. The Phillips Exeter Academy describes the Harkness method thus:

As in most Academy classes, mathematics is studied seminar-style, with students and instructor seated around a large table. This pedagogy demands that students be active contributors in class each day; they are expected to ask questions, to share their results with their classmates and to be prime movers of each day’s investigations. The benefit of such participation in the students’ study of mathematics is an enhanced ability to ask effective questions, to answer fellow students’ inquiries, and to critically assess and present their own work. The goal is that the students, not the teacher or a textbook, be the source of mathematical knowledge. (Exeter, 2016)
It is not the teacher but the students who are the “prime movers” of the “investigations,” so Harkness method definitely falls within the set of IBL methods as opposed to the traditional lectures in which teachers pose the questions and teachers decide how the students will solve the problem by demonstrating example solutions. Students might begin with a challenging question in class, and if stuck, teachers might help by offering leading questions, but never by suggesting a next step (Geary, 2013). The ban on giving away the next step is serious, and because no textbook is being followed, Harkness students will see “no cues in the form of chapter headings, or sample solutions that precede them. Instead, the problems require them to consider a question, and search their own ‘toolboxes’ for possible strategies.” (Geary, 2013) This moment of strategic choice is something Harkness shares with MMM, but certainly does not share with traditional textbook and lecture math. As students are doing the work, the teacher has the delicate task of judging when to let the struggle continue and when to offer a leading question, so while Harkness certainly can be described with its practices, it is neither formulaic nor repetitive, but rather “messy” (Geary, 2013). The role of a Harkness teacher seems more like a leader in an improvisational band, not necessarily dictating what chords or notes the musicians will play, but still making it all come together as beautiful music, not noise. Of course, the admission that it can be “messy” means that wrong notes will happen in this improvisation. But the musician who can play the same songs with little variation is not as skilled as the musician who can play around with the tune, introduce new modes, new chord progressions, new solos.
Teachers delivering content...miss the mark. Our students are able to be much more active learners and thinkers, if we [teachers] can step away from the front of the classroom and put interesting and engaging problems before them instead. It is our responsibility to empower students to think critically, creatively, and collaboratively to solve meaningful problems on their own. It is possible to do so without sacrificing content. But the value of such an outcome transcends content, and trumps traditional teaching and delivery. We should expect more from our students, and they should expect nothing less from us. (Geary, 2013)

By emphasizing that the students can learn to answer mathematical problems “on their own” and that this is more valuable than “content” or “delivery,” these proponents of the Harkness method are making a very similar assertion to R.L. Moore’s assertion that building students’ “strength” or “power” was more important than giving them “information” or “knowledge.”

The Interactive Mathematics Program

There is another high school curriculum which shares features with Harkness math and the MMM: the Interactive Mathematics Program (IMP). IMP is an alternative curriculum funded by NSF, which reordered and reselected topics in high school mathematics, so topics that would traditionally be in a Geometry course are spread over four years, and so, too, is it with Algebra 1 (IMP Northwest, 1997). Problems, as opposed to topics, are front and center. Students learn about basic trig functions in a chapter titled “The Return of the Tree” as opposed to most geometry textbooks which would have titled the chapter “Right Triangle Trigonometry.” The skill of manipulating linear equations and
solving systems of equations within the field of linear programming is taught in a lesson entitled “Cookies” (Math IMP, 2012). IMP asks fewer questions, and encourages more discussion. An IMP classroom is described on their website as having:

- An expanded role for the teacher
- A more active role for the student
- Extensive oral and written communication by students
- Both teamwork and independence for students
- Assessment using a variety of criteria (IMP)

And when these bullet points are expounded upon, IMP clearly shares similarities with MMM and Harkness, in that the teacher is not providing examples but instead coaxing ideas out.

The teacher asks challenging questions and provokes students to do their own thinking, to make generalizations, to discern connections and relationships, and to go beyond the immediate problem by asking themselves ‘What if?’ The teacher uses his or her expertise to provide the ‘glue’ needed to help students tie ideas together and to clarify any misconceptions that may arise. (IMP)

As we can see, this teaching philosophy is about building “strength” (as R.L. Moore called it) as well as knowledge:

In many traditional classrooms, a student's task is to mimic the work presented by the teacher and to find numerical answers to similar problems. But in a world that is ever changing, students need to be equipped to handle problems they have never seen before, and to handle them with confidence and perseverance.

To meet this need, the IMP curriculum is designed to give students a more active part in their learning. They work with complex and
realistic situations, rather than with problems fitting a rigid format. They construct new ideas by moving from specific examples to general principles. They progress beyond simply finding numerical answers; they use those answers to make decisions about real-life problem situations. They generate probing questions for each other and challenge each other's ideas. They must justify their reasoning by explaining to the teacher and to their peers what approaches they tried, what worked, and what didn’t. (IMP)

IMP contrasts itself with a traditional method that involves repetition of knowledge by the use of the word “mimic.” This is reminiscent of R.L. Moore’s critical comment about knowing the definition of a function: “Who told you and why did you believe him? Because you thought it was the proper thing to do?” (Parker, 2005, p. 261) IMP, like Moore, aims for students to explore the foundations and logic underlying mathematical knowledge. When students “justify their reasoning” and “challenge each other’s ideas,” that is similar to the presentation and critique of proofs by students in MMM. A common attitude among proponents of IBL is that traditional lecture and textbook methods offer a low-energy equilibrium state for math students, wherein mimicry can generate correct answers and deeper thinking can be avoided or missed entirely by students. Proponents of IBL aim to pose questions and problems in such a way that students must engage in deeper thinking, creative tactics, strategic choices, and critiques.

While IMP and MMM are both examples of IBL, it is worth noting a difference between the methods in that the Moore Method (and to a lesser extent MMM)
individualism and independence seem to be emphasized over collective work. IMP touts itself as teaching teamwork and independence.

IMP students spend much of their in-class time working together in teams; the curriculum promotes this type of interactive learning through its use of complex problems. In or out of class, they are encouraged to talk and do mathematics with other students, with teachers, and with parents. They learn to share ideas, build on each other's efforts, communicate, and take risks.

At the same time as students are expanding their ability to work productively with each other, they are also gaining independence as learners and thinkers. Because the curriculum demands a more active role from them, they demand more from themselves. The result is a classroom in which students take individual and joint responsibility for their own learning. (IMP)

The Moore Method and MMM discourage early collaboration even out of class, but IMP explicitly desires that students talk with each other and with people who are not students in the class. Thus, IMP takes the more collaborative approach, as does Harkness.

Because the Harkness method and IMP are designed for secondary school students, and MMM and the Moore Method are for undergraduate and graduate students, this difference is understandable. It is generally expected in America that post-secondary students have more autonomy in their education than their younger counterparts in secondary school, and it is reasonable for a teacher to think that in a secondary school to deny students collaboration on mathematics could be less productive.

From these four examples of inquiry-based mathematics curricula (and doubtless, one can find other forms of IBL) we can conclude that proponents of IBL see their work
as distinct from and superior to traditional lecture methods, and believe IBL offers more valuable experiences than the traditional offers. Students in IBL are afforded more opportunities for deeper thinking, creativity in solving problems, critical thinking, proving conjectures, and questioning ideas. These are attractive points, especially to teachers who have themselves enjoyed the moment of realization after struggling with a tough problem, of achieving a result that shows a surprising connection between ideas. Many mathematics teachers who have not been exposed to IBL themselves will be attracted to using such methods as they learn more about them. One question that arises as a secondary school teacher weighs such a choice of methods is “What will be the effects of IBL on students?”

Studies on Effectiveness of IBL

In general, a growing body of research suggests student gains from IBL, and these gains are not limited to mathematical thinking, but include improved student attitudes about mathematics. For example, “The effects of IBL include benefits for motivation, for better understanding of mathematics, and for the development of beliefs about mathematics as well as for the relevance of mathematics for life and society (Bruder, 2013).” Some studies did research the benefits of IMP when it was implemented with students (Merlino, 2001). In Philadelphia in the 1990s, IMP was implemented side-by-side with traditional math courses, and afterward scholars retrieved data with student attitudinal surveys, PSAT scores, passing rates, and SAT-9 scores. The attitudinal surveys
measured whether students wanted to continue with IMP or switch out, and in each school, the response was a majority wanted to continue with IMP—in some cases as high as 90% (Roberts, 1995). In addition to students preferring IMP over Algebra courses, IMP students had higher passing rates and attendance rates, and higher scores on both math and verbal portions of the PSAT in 10th and 11th grade (Merlino, 2001). So, the attitudinal surveys seem to indicate that this inquiry-based learning increased student interest in mathematics, and the score data seem to lend credence to the idea to that inquiry-based learning effectively improves students ability to think mathematically.

While claims have been made that IMP has greater “effectiveness” than traditional instruction at improving student performance on exams, one best-evidence synthesis written by Slavin, Lake, and Groff (2009) considered the methods of comparison in each of these articles, concluding there is insufficient evidence to claim IMP is more effective than traditional methods of instruction (Slavin, 2009). Rigorous scientific comparison studies are difficult to perform in the field of education because of the many factors that could influence the control and variables. The best-evidence synthesis noted that the Merlino (2001) study on IMP was not begun with a randomized selection of students but rather the data were examined post hoc, with exclusions done to make IMP and non-IMP cohorts of students start from a comparable baseline. Because the computer scheduling which selected the students for IMP or non-IMP is subject to pressures when students select other classes like Honors English or Math Test-Prep, the
best evidence synthesis concluded there was insufficient evidence to claim IMP was more effective than transition instruction. However, in their best-evidence synthesis, Slavin, Lake, and Groff did conclude “programs that affect daily teaching practices and student interactions have more promise than those emphasizing textbooks or technology alone (Slavin, 2009).” This conclusion is encouraging for proponents of IBL because daily student interactions and the daily teaching practice of getting students to think on their own terms is part and parcel of inquiry-based learning.

In the next chapter we will discuss difficulties encountered with IBL, but we should address how teachers must make decisions about what and how to teach with the information at hand. The evidence about IMP from Merlino (2001) and other studies cannot be interpreted to say that IMP is worse than traditional instruction on student achievement. Thus, it can be concluded that it is very unlikely a teacher would be doing a disservice by choosing IBL, and—as we noted earlier—the best evidence synthesis did conclude “programs that affect daily teaching practices and student interactions have more promise than those emphasizing textbooks or technology alone,” (Slavin, 2009) which leads one to think IBL is a good bet, even if the evidence is imperfect.

There have been some studies with significant implications for IBL, though they focus on college courses rather than secondary education. For the purpose of comparison with lecture courses, the characteristics of an IBL math course could be considered to be:

• learning goals focused on problem-solving and communication
• a curriculum driven by a carefully constructed sequence of problems or proofs,
• driving toward a small number of “big ideas”
• course pace set by students’ progress through this sequence
• class time used for a mix of active and collaborative problem-solving tasks
• instructors who guided student work instead of delivering information. (Laursen, 2013)

These characteristics resemble the Harkness Method, MMM, and IMP as described earlier in the thesis. Lecture methods were compared against IBL (called “active learning”) in mathematics courses across four institutions of higher learning, and the following conclusions were presented with a dig at traditional lecture pedagogy.

College instructors using student-centered methods in the classroom are often called upon to provide evidence in support of the educational benefits of their approach—an irony, given that traditional lecture approaches have seldom undergone similar evidence-based scrutiny. Our study indicates that the benefits of active learning experiences may be lasting and significant for some student groups, with no harm done to others. Importantly, ‘covering’ less material in inquiry-based sections had no negative effect on students’ later performance in the [mathematics] major. (Kogan and Laursen, 2014)

In summary, this study has found a way to benefit some students without negative effects on others, and noted that the decrease in coverage (a critique commonly made by detractors of IBL) did not have discernible impacts on student work in later courses. So if the negative effects do not show up, but the IBL method benefits some students in a “significant” and “lasting” fashion, the case for IBL over lecture is compelling. We should be clear about which students experienced these benefits and about the magnitude
of the benefits. The study disaggregated the students into three groups: low achieving (GPA < 2.5), medium (2.5 < GPA < 3.4), and high (3.4 < GPA), then compared their performance in math courses over time.

Taking an IBL course did not erase achievement differences among students, but did flatten them. In non-IBL courses, initial patterns of achievement difference were preserved; previously low-achieving students gained no ground. (Kogan, 2014)

This shows that taking an IBL course was no miracle cure for achievement gaps but reduction in the gaps was observed, while those achievement gaps persisted with students in the lecture-style math courses. The study also observed that when women took a lower-level undergraduate IBL course like multivariable calculus, the experience of IBL correlated with women taking on average one more elective math course in later semesters as compared to women who took a comparable non-IBL course. This correlation indicates that IBL builds enthusiasm for mathematics with women, an important effect which could be used to bring more gender balance to the male-dominated field of mathematics. Earlier in the thesis, it was noted that the NSF considered it troubling that curriculum was boring students out of STEM classes, but now there is data that the method of IBL builds student interest (as well as performance) in math, helping to solve the problem pointed out by the NSF and Lockhart, among others.

Knowing that IBL offers advantages over traditional methods, the question arises: is it possible for teachers to make the transition to IBL and to improve over time to greater proficiency with IBL? There is hopeful evidence in the literature that the very act of teaching in a classroom with IBL is a form of professional development (sometimes
called Learning Through Teaching) in which a teacher’s proficiency improves in measurable ways (Leikin, 2006).

For a teacher who is considering the option of incorporating IBL practices into the class, the preponderance of the evidence so far weighs in favor of choosing IBL over traditional direct instruction. Studies with a large N of students (Laursen, 2013, 2014) show IBL has improved students’ performance (especially among low-achieving math students) and students’ interest (especially among women) in taking more math courses at the college level with no harm done to other groups of students. Studies have shown gains in learning through IMP (Merlino, 2001) and surveys show IMP boosted student interest (Roberts, 1995). Moreover, IMP contains the sort of daily practice and student interaction that boosts student achievement according to the most rigorous standards of educational research (Slavin, 2009). The picture becomes clear of how direct instruction as a delivery mechanism for information can bore students and give them the impression of math as a tremendous set of rules to be memorized, reducing student interest in math. However, inquiry-based learning offers an improved educational experience with mathematics by stimulating student interest and learning. We now turn to how a secondary school geometry teacher could implement inquiry-based learning practices into her or his own classroom.
The previous chapter makes the case for secondary school teachers to transition to more inquiry-based learning in mathematics. However, before a teacher begins replacing curriculum, it is wise to consider what challenges the transition may have, to anticipate barriers to entry into IBL teaching. As we have seen in the first chapter’s history of IBL, inquiry-based learning has been around a long time and has been tried in different times and at different levels of educations, such as the undergraduate level versus the secondary school level. The nature of secondary school mathematics courses differs from that of undergraduate mathematics courses, so if a plan to transition to inquiry-based learning is to be successful, questions should be asked about IBL as it has been actually practiced in secondary schools. To understand how a transition can happen and what can impede the transition, research interviews were done with seven current and retired mathematics teachers experienced with IBL at the secondary level. Among the interviewees are secondary school mathematics teachers from private and public schools in the United States. The interviewees reflected on their experiences with IBL in a useful fashion for teachers who wish to make the transition. As we consider the content of these interviews we will observe that some of the challenges of using IBL today are very similar to tensions between IBL and traditional lecture in distant history. Some barriers are raised by people such as students, teachers, administration, guidance counselors, and parents associated with the institutions, while other barriers arise related to resources such as time, curriculum books, and professional development.
The interview questions are listed below. Full individual responses may be available upon request, though the privacy and identity of the respondent will be protected. For the sake of brevity, this thesis will only use the parts of responses that are relevant to questions considered in this thesis.

1. What do you think distinguishes a class taught in the IBL mode as compared to a class taught in a more traditional mode?

2. Have you ever been a student in IBL classes? Which course(s)? Where?

3. Consider your experience in the IBL class. How did it differ from non-IBL courses you took?

4. Did you gain something or learn something different from this IBL course than you did from non-IBL courses? Can you describe a distinct effect IBL had, if any?

5. Have you ever taught an IBL course or lesson? Describe it.

6. How did students react to the IBL method? Did your students gain something or learn something different from this IBL course than they did from non-IBL courses? Can you describe a distinct effect IBL had, if any?

7. Did your administrators react differently to the IBL method? What sort of feedback did they give you about IBL?

8. Did the parents of your students give you any feedback about IBL? Describe the feedback.

9. How did lesson planning differ for the IBL course?
10. How did your method of grading change for the IBL course?

11. What sort of barriers would teacher have to face to begin teaching IBL?

What could be done to pass these barriers?

These questions were chosen to examine how teachers’ minds shifted from one form of education to another. Questions 1, 2, 3, and 5 help the researcher to be certain that what the interviewee calls IBL matches what the researcher thinks of as IBL. Questions 4 and 6 address what the history of IBL tells about how IBL provides learning beyond mere facts or mimicry of procedures. Questions 7 through 10 explore the pitfalls and challenges. A sample of responses will now follow.

Of the seven teachers interviewed all described an IBL course in mathematics in terms very similar to this thesis description of IBL. Therefore we are confident the responses regarding factors involved in the transition will be relevant and helpful to teachers who wish to make the transition themselves. Three sample responses to Question 1 follow.

Teacher No. 7: “From the teacher point of view, the traditional has more explaining, teacher talk in an attempt to explain. In IBL, more student talk, more variety to solutions. Problems are different, more open-ended questions."

Teacher No. 5: “An inquiry and discovery approach would involve more strategies that allow students to figure things out on their own. There would be more inductive reasoning and looking for patterns. The teacher, and class materials, would guide the student to discover. The traditional mode would involve more direct instruction. The teacher would explain more, and the focus would be more on the teacher and the material.”
Teacher No. 1: “[In an IBL classroom] The teacher is a participant. Many correct processes [are allowed.] Students [are] observed to be thinking. Students are conversing with other students, listening, speaking. Strategic arrangement of room [in] groups of 4.”

These are typical of the other responses. Like the Harkness method and IMP, these responses show students engaged in more autonomous approaches to mathematical thinking, as opposed to the tradition of following the math teacher’s directions. After confirming the consistency of understanding what IBL is, the interviews can be used as a source of information about how a transition to IBL can succeed. To succeed, a teacher will have to pass certain barriers.

The Barrier of Standardized Testing

The first barrier to consider is the tension between IBL and standardization. Schools and universities exist as as bureaucratic entities that must follow demands and regulations from administrators locally, who in turn are following perceived directions from politicians. A common belief in the 20th and 21st centuries is that schools or universities can better guarantee quality in their services through standardized methods. For secondary schools in the United States, this means standardized testing, such as the PSAT, SAT, ACT, MCAS, etc. Often public high schools have standardized departmental exams for individual courses. Private schools, too, experience pressures for standardized testing as their students may need to take college entrance exams such as the SAT. It is common for administrators to believe that the best way to guarantee achievement by students on standardized exams is to standardize instructions across courses within
schools or within districts. Thus the traditional form of math instruction relying on lecturing content seems to be a good strategy to meet the demand of preparing students for a standardized exam, because it seems to cover a predictable and large amount of content. Because a high school geometry class must prepare students for a standardized exam, such as the MCAS in Massachusetts, the teacher will feel pressure to avoid IBL. From an interview with Teacher No. 6 we note this statement: "I see time and the pressure on teachers to have their students perform on high-stakes standardized assessments as barriers.” A similar sentiment may be found in the history of R.L. Moore, who taught at Princeton early in his career, and was assigned as preceptor to an introductory calculus class with a final exam standardized across the different sections of the course and a required standardized curriculum. Moore was not happy about this and “apparently believed this would merely encourage examination-driven teaching at the expense of developing a student’s overall ability (Parker, 2005, p. 84).” So even a hundred years ago, the tension between standardized testing and IBL existed. While today’s proponents of IBL can point to research studies, such studies are not popularly known, and standardized testing in secondary schools holds a power today that is orders of magnitude greater than in the early days of R.L. Moore’s career at the university level. Therefore, a teacher who transitions to IBL should have some background knowledge of how IBL improves student achievement and student interest in mathematics, and the teacher should be ready to communicate those points when needed. It is anticipated that as IBL becomes more prevalent in schooling, standardization, including standardized assessments, may need to accommodate IBL.
The word “communicate” implies more than one person is involved. Other people are involved in education because a teacher needs students, who will necessarily have parents or guardians who desire what is best for their children, and the parents or guardians will communicate with administrators who have decision-making authority within the school. The thesis will examine resistance to IBL first among students, then among parents, and lastly, among administrators from evidence gathered in interviews with seven teachers.

**Barriers in Students’ Attitudes**

From teacher interviews, it is clear not every student will be on board with moving more toward IBL in the classroom. (It should be noted that with direct instruction, one may observe students being disengaged or frustrated as well). Almost every teacher interviewed noted some form of student resistance to IBL. The second teacher commented, “It was not how they [the high school students] felt math should be taught; they wanted me to lecture, show them an example, do an example with them, and then give them 20 practice problems just like the one we tried.” In describing this resistance, the teacher illustrates how lecture and practice exercises are considered the default mode in secondary education by students, who may ask for it instead when faced with inquiry-based learning. Similarly, Teacher No. 7 described student resistance in this way: “Kids resist the work of open-ended problems, and they are often the high-achieving kids, who want to practice recipes with product answers, not process answers.” Teacher No. 7 uses illuminating terms to contrast IBL with lecture and practice.
“Recipes” are explicit steps given to the students, not steps deduced by the students’ own exploration and inquiry. “Product answers” are similar answers like R.L. Moore’s example of the memorized definition of a function. “Process answers” would be responses that describe how and why a student knows an idea is true. Teacher No. 7’s mention of high-achieving kids recalls the earlier Lockhart (2002) quote about students who arrive at graduate school thinking they are “good at math” only to find they cannot think creatively. Almost every teacher surveyed had observed resistance from students, but some interview responses included a positive change in that student attitude. Teacher No. 4 found that this resistance did not last forever: “Some hated [IBL] at first, some loved it at first. Generally speaking, they all got on board by the end of the first quarter, and many of those who didn’t like it at the start really appreciated it as the year went by.” From this comment, it can be gleaned that patience and persistence with the IBL approach can be part of a successful transition. Teacher No. 4 expounded further on student resistance and explicitly connected the “lecture-as-default” belief to the resistance:

The culture-clash that is the reality of math as a creative, sometimes collaborative, thing that humans do and the vacuous and robotic way that we typically teach the masses to ‘learn’ math. (see Paul Lockhart, A Mathematician’s Lament.) Getting through it is: 1. Having faith that you’re right, 2. facing the clash, maintaining faith, and relying on that to be resilient. It can take a couple of months to get everyone on board, but it is worth it.

These words speak to a great difficulty, and there is a bit of the missionary ethic in these words. For Teacher No. 4, the end of the story is happy because the teacher
manages to “get everyone on board.” The effort of swimming against the current is “worth it.” Almost all of the interviewed teachers had ultimately positive experiences with IBL.

Yet one interview stands out. The third teacher’s experience in using a curriculum close to IBL in a high school geometry course illustrates how students’ reactions can be so strong that the transition retreats to more of a lecture style of education.

Some students experienced a lot of frustration, as they were not used to having to work through material. They wanted the information to be handed to them. This meant a lot of blowback from parents and administration. Students claimed that I was NOT teaching and wouldn’t help them when they had a clear set of steps to gain assistance. The point came where administration felt that they could not justify continuing the IBL structure of the course. This then lead to quite a fight between administration and the mathematics department. In the end I was not willing to continue the battle so I gave the Honors Geometry to another member of my team, and he does some IBL but not a full curriculum as I had.

This is a serious challenge. Student resistance led to parent resistance, followed by administration resistance, which discontinued the IBL nature for much of the course, returning it to more of a lecture-based course. We can conclude that if a transition is begun, there is substantial chance for the transition to fail, and the default mode of lecturing and practice to continue its dominance. Later in the thesis, methods for avoiding this kind of situation will be addressed. While the experience of Teacher No. 3 seems especially disappointing, the preponderance of the experiences of the interviewed teachers do offer hope and strategies for overcoming the barriers. The fifth teacher noticed students did adjust to the more inquiry-based approach, saying, “Now that we’ve
moved to Common Core for a few years, students are becoming more used to inquiry methods of learning, collaboration, and communicating about the math.” The sixth teacher likewise found that with time, students in the detractors-to-IBL camp altered their attitudes, answering:

Many students with an aptitude, interest and/or confidence as math students often reacted positively. Many students with an aptitude, interest and/or confidence as math students, were uncomfortable with this, just wanted ‘show me what to do,’ just wanted an A. Students with a low aptitude, interest and/or confidence as math students, usually were resistant, so persistence patience on my part was necessary, but I did not abandon IBL, but made accommodations and adjustments as needed (as always.)

This story from the sixth teacher illustrates that resistance to the method may appear both in students who have a history of excellence in traditional methods as well as students who have histories one might not describe as excellent in math. Whether these students with a not-so-excellent history in math courses suffered from math anxiety, skill deficits, or lack of interest, they were still required by law to take geometry in high school, and one might even speculate they would not appear in such a class without state requirements. The sixth teacher persevered and adjusted to maintain inquiry methods in their education. We note that when a teacher adjusts or accommodates practices to educate with inquiry, this reflects how a student’s mind in inquiry must adjust or adapt to challenges in understanding a substantial math problem. Like solving an open-ended problem in mathematics, bringing students “on board” with a transition to IBL can require adjustments and new thinking from the teacher.
Barriers in Adults’ Attitudes

In addition to addressing the possibility of student resistance to IBL, teachers must also consider resistance from the other adults involved in education: parents, guidance counselors, and administrators. We have already seen, in the story of Teacher No. 3, that if two or three of these groups align against IBL, it can induce the teacher to return to direct instruction. As we continue to examine the interviews of teachers with IBL, we find that a supportive administration was helpful to making the transition to IBL. When asked if administration supported IBL, Teacher No. 6, who had more than 20 years of experience, replied, “This varied depending on the administrator. Most feedback was generally positive.” Teacher No. 4 described administrators who fully approved: “They love it. They want everything to be student-centered, so I introduced this at a time when School really received it well.” These administrators may not understand the nature of mathematics, nor the value of IBL in producing better research mathematicians down the line in graduate school, but their interest included student-centered education, and they saw how IBL fit that characteristic much better than lecture did. Likewise, Teacher No. 2 had administrative support.

My administrators fully encourage me to use IBL. They also want to see a classroom in which every student feels confident as they actively engage in learning math...[IBL] requires full support of school leaders. I could not do what I do without knowing that I have the unconditional support of my director!

The phrase “actively engage” here allows us to infer that these administrators understand the nature of “active learning,” of which IBL is a part but direct instruction is...
not. The preponderance of evidence from interviews leads to the conclusion that a teacher who wishes to transition to IBL would be well-advised to communicate the value and advantages of IBL to administration to earn and maintain their support for this method over the traditional lecture style of mathematics classes.

Beyond the teacher and the administrators we find another influential set of people in a secondary school. The guidance counselors help students to choose math courses and act as conduits of information to students about what colleges expect and what the nature of a good mathematical education is. In Teacher No. 7’s school, the IBL math courses did not have an “Honors” designation. This created a bimodal distribution in the guidance counselors. Some guidance counselors thought that without “honors” math a student would have less chance at acceptance to a selective college. This group of guidance counselors would not even pose the chance to think about choosing IBL versus traditional math courses when talking with students about their schedules. Another set of guidance counselors were “key to making it work” because they did give the students a choice of IBL. Among the interviewees, only teacher No. 7 mentioned the guidance counselors, perhaps because the interview questions did not specifically refer to them. We conclude that for a teacher who wishes to use IBL, it will be important to be communicate with guidance counselors so the counselor understand the value of IBL and its advantages over traditional lecture. When one considers the argument that “honors” math is necessary to enter a selective college, it could be a good counter-argument that Phillips Exeter Academy’s Harkness Math is a form of IBL, and PEA students have a
good reputation for entering selective colleges. As with administrators, guidance counselor buy-in can aid the transition to IBL.

The remaining group of adults who can act as barriers to entry are the parents of the students. Obviously, parents will want to be informed about their children’s education and to be helpful, and they will advocate for better education for their child. Many parent reactions are possible. Teacher No. 4 saw the parent reactions as falling in three categories: “Parents are either thrilled that their kids are finally being ‘seen’ rather than only graded on their tests and quizzes, annoyed that they are not being taught the material (even though they are), or are cautiously optimistic that it could all still be ok in this environment.” The first category of parental reaction is a positive one made possible by the way IBL opens the math classroom to individual student expression and creativity. The second reaction category is a product of the traditional way to teach mathematics, which critics have often called ‘mimicry’ because it does not recognize that inquiry is a form of teaching. The third category is more of a neutral judgment on the parents’ part. A teacher would be well-advised to pay attention in conferences to which parents have the negative or neutral reactions so as to make an effort to assure those parents their students are making progress. We have already seen, from the interview with Teacher No. 3, how parental fears about IBL can cause a retreat to traditional lectures. There is more to Teacher No. 3’s story, however, showing positive side:

Students who found success loved it, and praised the method, and so the parents did as well. Students who were used to being handed everything and never having to struggle, gave negative feedback to parents like that they were not being taught, or they didn’t
understand or why isn’t there another option because my kid isn’t that type of learner.

So, in the school where Teacher No. 3 used an IBL style of mathematics, there were students who “loved it,” and that attitude transferred to their parents. So there was the potential for positive parent feedback to support the IBL approach. The other, dissatisfied students we know convinced the administration to pressure the course into changing. A further question reveals that Teacher No. 3 has reflected on the events:

Question: What sort of barriers would teacher have to face to begin teaching IBL? What could be done to pass these barriers?

Teacher No. 3: Making sure that the department and administration is behind the teacher. It would also be important to make sure parents are aware of the reasoning, resources and support available to students who might struggle with this type of learning.

The experience of teacher number three illustrates the importance of setting the stage for the transition to begin and to succeed. By enlisting the support of the department administration with inquiry-based learning, a teacher helps to buy a little more patience as students adjust from direct instruction to inquiry-based learning. Parents will want to understand why inquiry-based learning is a better method so it will help for a teacher to be ready with information about studies showing the success of inquiry-based learning. In addition to the studies mentioned in our chapter on the history of IDL, there is work, such that by Carol Dweck (2006) and Jo Boaler (2013), which shows the importance of the growth mindset in math education. If we look at the wording from teacher number three, some of the parents of dissatisfied students said their child wasn't "that type of learner."
The phrase "type of learner" allows us to infer that some of these dissatisfied students possessed a fixed mindset—that is, a belief that ability in mathematics is something a person cannot change, and struggle is a sign of stupidity. However, a growth mindset can be developed in students, and this growth mindset can improve students attitudes about learning mathematics, that learning is a matter of a fruitful struggle. Teaching students about this growth mindset can give them hope that when they struggle with mathematics that is a sign of learning, not a sign of stupidity. In this way, the teaching of a growth mindset can prevent parent concerns about struggling students from becoming parental resistance to IBL. The lesson of the story of Teacher No. 3 can best be found in a pithy observation from Teacher No. 7: “Addressing the fears of adults is an inevitable piece of the puzzle.”

Another piece of the puzzle, for a teacher who wishes to transition from lecture to IBL, is to have the proper changes to practices and procedures. Interviews with teachers who have experience in the IBL approach have noted several common points of changing procedures. An IBL course will require significant changes to the problems given to students, to the lessons planned, and to the assessments and grading systems. A common point among teachers who were interviewed is that more emphasis (in grading) had to be placed on participation in class (in one case a teacher advocated participation be 40% of the grade) and more emphasis on qualitative assessments, though all teacher still used quantitative assessment such as quizzes. For the reader who plans to begin a transition to IBL, a later chapter of this thesis will provide sample lessons and guidelines to create lessons in the IBL style.
The three procedures mentioned—a curriculum of good problems, good lesson planning, changed grading systems—may all be considered something to be dealt with as professional development (PD). Teacher No. 6 reported,

I also see a common teachers’ practice of teaching as they were taught in a traditional classroom, and a lack of training in teacher prep or PD opportunities in IBL as barriers. To overcome the latter of these barriers, I’d recommend some sustained PD in IBL methods, such as study groups for teachers to experience IBL as learners and/or Lesson study experiences focused on implementing IBL practices.

The idea of training for IBL among teachers is important. IBL is a trickier method to use than lecture and presentation, and because most math teachers were taught by lecture, it can be difficult to switch out of a lecture perspective. Study groups of teachers at a school comparing lessons and offering improvements would benefit practitioners of IBL by offering social support. One might consider the proposal of this thesis as an encouragement for a math teacher to go it alone on IBL, but if a teacher can find colleagues who will join in, that is preferable. Teacher No. 7 offered this perspective on collaboration among teachers: “Unless there’s a shared goal, [teaching IBL] can feel isolating. The opportunity to troubleshoot helps, and it is important to do this with other people.” Lesson study experiences, wherein teachers compare different approaches and tactics within a lesson are what this teacher means by “the opportunity to troubleshoot.” Teacher No. 7 had been through training to teach an IBL course, and admitted feeling “nervous about the switch [away from lecture] but six of us [teachers] went into this together, had time together.” This social support proved helpful in the estimation of
Teacher No. 7. The need for training and collaboration among teachers was echoed by Teacher No. 2 as well:

It takes patience because the reality is it is more than teaching IBL. It is shifting the entire culture of what it means to do math, learn math and teach math. I think we need to really be considering how we structure our schools to build a community of learners among the teachers. Without this, our teachers and students will never understand what it means to collaboratively learn.

The insights offered here take the idea that IBL teachers should collaborate and push it into a broader context of “shifting an entire culture” to emphasize that a transition to IBL will be a much more consequential change than the adoption of a different textbook. Mindsets of students will be changed, and the mindsets of teachers will be changed, too. To give an example of how the culture of teaching math would change, Teacher No. 7 explained how teaching with IBL “made me see thing differently, made me realize finding a solution isn’t a stopping place. The goal is to come up with a number of different solutions…I would pay attention to the way others see things.” This quote gets at the nature of IBL work because a teacher who transitions away from the traditional way of teaching math will begin to quit looking for student responses that match the answers in the Teacher Edition, and instead start looking a student responses for evidence of how the student is thinking about the problem.

The evidence gathered in the interviews of these current and retired math teachers does paint a picture of significant barriers to be overcome when a teacher makes the switch to IBL, but it is possible to move beyond these barriers or to deal with any resistance that may arise. This brings the thesis to Chapter 4, the sample lessons.
Chapter 4  Proposed IBL Lessons

We have seen a problem in the outcome of traditional education of students following directions unthinkingly when presented with mathematical questions, and in the literature we have seen a strong argument for more teachers to use IBL to improve students’ performance and interest in mathematics. Interviews with secondary school math teachers corroborate the literature’s promise and the possibility of transitioning to IBL. Can a teacher make their own transition to IBL from the traditional lecture style? This thesis proposes that it is not only desirable, but also possible. To that end, sample lessons in the inquiry-style will be presented, followed by guidelines on how a teacher may take an existing curriculum and transform it to offer more inquiry for students. This chapter offers three lessons which a teacher may use in a secondary school geometry course. The first will be a structured inquiry (or structured discovery) and the next two will be guided inquiry lessons. This chapter will include scripts and advice (also known as “teacher notes”) to guide the teaching of the lesson. The following chapter will provide guidelines on how a teacher may create such lessons themselves.

A warning to the reader: If this is the first time you are reading this, perhaps it would be better to ignore all the prose in this chapter and simply focus your attention on Figures 4-1, 4-2, 4-3, etc in this chapter. Consider them puzzles for you to solve, and cover up any visible prose with blank paper so that you might have the delightful experience of inquiring into mathematics, making your own observations and

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conjectures, and proving what you think. Then compare what you thought to what the author of thesis explains about these lessons. Reading Chapter 3 before you attempt the inquiry lessons could be like watching someone else unwrap your gift and play with it.

On the following page is Figure 4-1, A Worksheet in Structured Inquiry Style
1. With your partner, find the values for $w$, $u$, and $t$. Leave your calculations on the paper because the reasoning is more important than just having the right value.

2. These angles $w$, $u$, and $t$ are called exterior angles. (Why?) How did you calculate them?

3. Look at the left triangle. Think about the 2 given interior angles and your result for $w$. There is a connection there. What is it?

4. Look at the middle triangle. The interior and exterior angles are different, but is the same connection there?

5. Does the obtuse triangle follow the same relationship?

6. What is $m\angle NOP$?

7. To see why this is true, sketch a line through $O$ and parallel to $MN$. Look for a pair of corresponding angles and a pair of alternate interior angles. Mark the congruent angles.

8. Summarize: The exterior angle of a triangle is _______________________

Figure 4-1 Structured Inquiry Worksheet
The Structured Inquiry of the Remote Exterior Angle

The objective of the first lesson is to allow students to discover (or realize) that in a plane, the exterior angle on one vertex of a triangle has a measure equal to the sum of the two interior angles far from it, often known as the “Remote Exterior Angle” theorem. The proper placement of the lesson in the sequence of a course is after students have learned that the three interior angles of a triangle sum to 180 degrees, and that the two angles of a linear pair sum to 180 degrees. The worksheet should be presented with an introduction, and the following script is suggested.

Teacher: “I have a new question for you today. It is about the exterior angle of a triangle. If you look at the top of the worksheet I have passed out, there are some missing angles for you to calculate. After you calculate them, you (or we) are going to find an idea inside these examples, and then prove the idea is true for any triangle.”

The teacher should expect that students might work out these examples in multiple steps, then prompts student to discover (or realize) that their work in the example may be done with one step, saying: “I see you have found the exterior angles. How many steps did you need? Could you find them in fewer steps?” When using this worksheet, the teacher may find some students can do without verbal prompts by merely thinking about the conceptual questions 3, 4, and 5. Then the lesson asks for students to generalize for any triangle by using variables a and b for the degree measures of the two remote exterior angles, first writing in algebra, then finally expressing what they have
learned in prose. By using numerical, algebraic, and verbal expressions of the same concept, the lesson helps students to solidify their understanding of the theorem without a lecture presenting the idea. If a teacher wishes to extend the lesson, he or she may ask questions such as “What are the bounds on the values for a and b?” to help students realize that either a or b could be obtuse but not both simultaneously. Another extension could be to ask the students “What previous theorems must be assumed to prove the Remote Exterior Angle Theorem?” to help make explicit the connection among ideas, and to illustrate how in geometry new theorems build on older theorems.

On the following pages, in Figures 4-2 and 4-3, we see our next example of a teacher-created, guided inquiry lesson on Transformations.
What transformations happened here? Can you use your compass and straightedge to figure out the details?
What transformations happened here? Can you use your compass & ruler to figure out the details?

Figure 4-3, Transformations Inquiry, page 2
The Transformations Inquiry

The second lesson, the Transformations Inquiry, is more in a style called “guided inquiry.” A question is posed to the students, but there are no intermediate questions as in the structured inquiry, no scaffolding immediately present. There can be scaffolding provided by the teacher to guide students forward but the teacher must wait and judge when is the right moment to offer questions or hints by observing students as they work with compasses and straightedges. The purpose of this lesson is for students to realize that compasses and straightedges (physical tools) in conjunction with perpendicular bisectors (mathematical tools) may be used to locate the mirror line between a pre-image and its reflection or the center of rotation about which a pre-image has been rotated. Students will see that construction tools may be used to uncover hidden lines or points. A teacher should offer this lesson after students have the skill of creating perpendicular bisectors with compasses and straightedges and understand the Perpendicular Bisector Theorem: any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

Upon distributing page 1 (see Figure 4-2), a teacher may ask the group, “What transformations do you recognize?” and take suggestions from students. Because there are many possible compositions of transformations that connect a pre-image with its image, the teacher should accept as many valid guesses as are given. Then the teacher may ask, “Is there a minimum number of transformations that map the first crown to its image?” This question is consistent with a broader theme in mathematics of finding
minima or minimal paths. It is expected that students will recognize that a reflection has occurred, and the teacher may ask, “If a reflection occurred, did it not have a mirror line? Where is the mirror line?” Give students time to figure it out. The desired conjecture here is that by connecting a point J and its image J-prime, then finding the perpendicular bisector, one has found the mirror line. Hints could be, “J went to J-prime, maybe they are connected? Could you draw a connection?” or “Maybe you should focus on just two points instead of the whole image,” or “Where is the mirror line between J and J-prime?” Students will construct segments from each original point to its image, and a hint could be “Does this segment J to J-prime help you make the mirror line?” Students then construct the perpendicular bisectors of the segments. As a final step in the lesson of proving the conjecture, students should cite the Perpendicular Bisector Theorem as they state that the perpendicular bisector of segment AA-prime is made of points equidistant from A and A-prime, and the mirror line of a reflection must be equidistant from each point and its image under the reflection. This realization paves the way for page 2 of the Transformations Inquiry.

Once students have used compasses to find the mirror line in a reflection, when the teacher asks, “What transformations happened?” on page 2, students should see this is not just another reflection. On page 2, the minimal transformation is one rotation about a single center equidistant from a point and its image. Of course, students may see that a translation happened followed by a rotation, or some other composition, and these are valid observations because any single transformation may be equivalently done by compositions of other transformations. Moreover, to see a composition of 2
transformations is not just extra trivia, it is a warm-up to a deeper geometric idea one may learn from this inquiry--at the end of this lesson, students may generalize that every rotation is equivalent to a composition of 2 reflections about non-parallel mirror lines. Guesses at the center may be asked for, as a way to engage students’ pride in their visual thinking skills and intuition about the problem.

After students recognize rotations, ask, “So if you know it is a rotation, can you find the center it rotated around?” If they stall, encourage them to draw the segments from points to images as with reflection. Note that these segments are NOT parallel, as before. Probably one person will draw a perpendicular bisector of AA-prime. (Note that in this thesis A-prime will be written A’.) If this yields no insight, say, “Why not try another perpendicular bisector?” This second perpendicular bisector will create an intersection point, which is the center we seek.

To explain further, the teacher should try to guide students to the following deeper understandings. The center of rotation is a point equidistant from each point and its image. So while there are infinite points equidistant from A and A-prime (a perpendicular bisector of them), and there are infinite points equidistant from B and B-prime (another perpendicular bisector), these two perpendicular bisectors will meet at a point, R. That point, R, is the center because R is equidistant from both pairs of points and images. AR=RA-prime, and you can construct an arc from A to A-prime to illustrate the rotation.

A lesson extension and a curiosity: Because a composition of two reflections about intersecting lines of symmetry is a rotation, that is why drawing two perpendicular
bisectors will find the point. And in that composition the measure of the angle of rotation is twice the measure of the angle between mirror lines. Beware the wrong notion that measuring the angle between perpendicular bisectors gives you the angle of rotation. The teacher should encourage students who conjecture this to verify it. If such a failure occurs then it is good experience to see a conjecture fail. And at that failure, remind them of the adage, “Good judgment comes from experience. Experience comes from bad judgment.”

Perpendicular bisectors are mathematical tools that can find points equidistant from other points. When carpenters cut a circular hole in the 2nd floor to install a spiral staircase, they need to know the center of the hole. They put two push brooms in the circle, letting the brush part be chords, so the handles will be diameters, intersecting at the center of the circle because the T shape of a push broom is a perpendicular bisector. The push brooms are a physical version of the mathematical tool, the perpendicular bisector. With inquiry, a teacher can encourage students see ideas like various theorems, definitions, algorithms, and shapes as tools to solve problems.

The third lesson, a teacher-created guided inquiry on Cyclic Polygons, now follows. It is the longest of the samples and consists of four pages to be distributed to students, though its purpose may be accomplished without doing all four.
Cyclic Polygons

This regular hexagon is cyclic because a circle can be constructed to intersect all its vertices. But some hexagons are not cyclic. Can you sketch a non-cyclic hexagon?

Irregular polygons are much less predictable than regular polygons. They offer us something to puzzle over: Which of these pentagons are cyclic? Which are non-cyclic? The duplicates are for extra trials.

Figure 4-4, Cyclic Polygons page 1
Cyclic Polygons

C)

D)
Cyclic Polygons

Which of these quadrilaterals are cyclic?

E)

F)

G)
Consider the special quadrilaterals such as the square, rectangle, rhombus, kite, trapezoid. Are any of these guaranteed to be cyclic? Or could some of them be cyclic if conditions are just right?

Figure 4-7, Cyclic Polygons page 4
The Cyclic Polygon or Not? Lesson Plan

Goal: The key question is “Which polygons are cyclic?” Student will inquire into methods to know which polygons are cyclic (i.e. can be inscribed in a circle), and by the end of the lesson should see that all triangles are cyclic. Logically it is equivalent to say that in all triangles, perpendicular bisectors are always concurrent, but not so with quads or pentagons. Some geometry courses teach that cyclic quads have supplementary opposing angles, and some teach the area of cyclic quad with Brahmagupta’s formula.

Assumptions: This lesson assumes students know how to construct circles, isosceles triangles, and perpendicular bisectors. It assumes students have seen the perpendicular bisector theorem. It assumes students have not constructed circumcircles of triangles.

Requirements: sheets with pentagons, quadrilaterals, triangles pre-printed for convenience; compasses; straightedges. Patty paper may be used to trace these pre-printed polygons.

Brief summary:

Begin with the question, which polygons are cyclic? First exploration is four pentagons. Next exploration is quadrilaterals. Last exploration is triangles. This should have a little “wow” because people will find out that every triangle is cyclic. This is one of those extraordinary properties of triangles that make them the “elements” of geometry.

Script:
Teacher: “I was once learning about cyclic quadrilaterals—those quadrilaterals which have all four vertices intersecting the same circle—and I wondered, could there be a way to know if a quadrilateral were cyclic without the diagram already having a circle? I wondered which other polygons are cyclic? So I made some pentagons. Let’s see which pentagons are cyclic…”

Distribute the pentagons paper (see Figures 4-4 and 4-5), give them some time to try their own approaches. If someone is already familiar with the circumcircle of a triangle, then they might bust through this early. But if they take time and need a hint, then ask “what do you need to know to construct just the right circle?” Look for them to say, “a center and a radius.” If they say they are looking for the center, then perhaps the question “Why is the center special?” or “What must be the relationship between the center of the circle and the vertices of the polygon?” to spark thoughts of equidistant points. Equidistant points should lead to the perpendicular bisector theorem (“Every point on the perpendicular bisector is equidistant from the two endpoints of the segment bisected.”) To be perfectly clear, one solution is to find perpendicular bisectors of all sides of the polygons. If the perpendicular bisectors of the sides are concurrent, then the point of concurrency will be equidistant from all vertices of the polygon, and a circle may be constructed.

Another route to the center might be the visualization of isosceles triangles. The center will have four or five radii extending to the vertices. When the center is found, the polygon interior will be filled with isosceles triangles. This is the same theorem in a different form because the perpendicular bisector of a segment is the locus of all vertices
of isosceles triangles with the segment as the base. Notice how this activity offers multiple paths to the goal of finding a center. The better inquiries offer students multiple paths to solutions, and thus have opportunity to discuss differing perspectives.

Though this script has just spelled out an algorithm, do not give it away. Perhaps they will roughly guess the correct center of the pentagons. Look for some students with good visual intuition to create circles that hit three or four of the vertices but not the last two or one. Encourage them. Give them the quadrilaterals page to work through as well until the majority of students is sure about which are cyclic and which are not. It is possible for an explorer to make a path through a forest and then at the destination, turn about to perceive the obvious trail in its entirety.

If the class gets to quadrilaterals without generalizing, the instructor may offer some form of this hint: “We need to find a single point (center) that is equidistant from all four vertices. Sometimes problems are tough because there are too many conditions to satisfy. So, we can solve an easier problem. Let us find a circle through just two vertices. Pick two and call them A and B. Mark an equidistant point from A and B. Is that the center? No? Mark another equidistant point from A and B.” At this stage, they might deduce that two points make a line, which could be the ignition spark for insight. If they stall after one line is drawn, have them pick another two points, by pointing out that since the choice of two vertices was arbitrary, one might choose another two and repeat the process of finding equidistant points.
Hopefully, this process of drawing perpendicular bisectors will be in use by end of the quads page. The teacher (or better, a student) could observe that if the perpendicular bisectors of all sides converge to a point of concurrency, then the polygon is cyclic. Also, students might construct two perpendicular bisectors, whose intersection will allow them to construct a circle hitting three vertices. Then, if the polygon is cyclic, this circle will intersect all vertices.

Among the special quadrilaterals, squares, rectangles, isosceles trapezoids are always cyclic. But only certain kites are cyclic if they have 2 right angles.

Another hint to move the lesson further: “What about triangles? There are different kinds. Could being acute or obtuse affect whether it is cyclic? Could being scalene or isosceles affect whether a triangle is cyclic?”

And then let them explore so they can see that all the perpendicular bisectors of any triangle will have a point of concurrency (circumcenter) and thus a circle maybe be circumscribed about the triangle. No such point of concurrency is guaranteed with polygons of more than 3 sides.

Extension of the lesson: It is possible to include the transitive property as in a proof that 3 perpendicular bisectors (lines $k,l,m$) of any triangle ABC must intersect at a point of concurrency. So any point on $k$ is equidistant from A and B, and if any point on line $l$ is equidistant from B and C, then their intersection (point O) must be equidistant from A and C by the transitive property [$OA=OB$, $OB=OC$, then $OA=OC$]. Moreover, because $OA=OC$, O must be on line $m$, the perpendicular bisector of segment AC.
Be ready to say “yes” to unpredicted conjectures and insights—and then have the students verify their conjectures, then prove or disprove them. This is the most valuable point of the lesson, to build curiosity and creativity in math.

Results of Student Surveys After Inquiry Lessons

We have seen in that studies of students who engage in IBL have higher interest in mathematics (Merlino, 2001 and Laursen, 2014), so the question arises as to whether these sample lessons, when given to high school geometry students will increase their curiosity. The Transformations Inquiry was given to two distinct geometry classes in a large high school, and the Cyclic Polygons or Not? Inquiry was taught to another geometry class in the same school. All three classes were typically taught with direct instruction. After experiencing the inquiry lesson, students were surveyed to compare the inquiry experience to compare their experience with inquiry to their experience with a traditional lesson. The survey asked students to rate their experience in this lesson on a scale from 1 to 6 over the following four continua.

(1) I explained less … I explained more than usual

(2) I asked fewer questions than usual … more questions

(3) I felt less curious than usual… more curious

(4) I understood less than usual the connections between ideas… more

Each of the continua had 1 for the left side choice and 6 for the right side. Averages are presented for each of the four survey items in each of the 3 lessons. In the
table below, 1T stands for the first time the Transformations inquiry was taught, 2T stands for the second time the transformations inquiry was taught, and CP stands for the Cyclic Polygons or Not? lesson. The lowest row represents the number of students completing surveys, which can be lower than the actual number of class participants as some chose not to complete surveys and some surveys were filled out incorrectly.

<table>
<thead>
<tr>
<th>Category</th>
<th>1T</th>
<th>2T</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explained</td>
<td>3</td>
<td>3.85</td>
<td>3.47</td>
</tr>
<tr>
<td>Questions</td>
<td>3.13</td>
<td>3.69</td>
<td>3.35</td>
</tr>
<tr>
<td>Curious</td>
<td>3.25</td>
<td>4.15</td>
<td>3.88</td>
</tr>
<tr>
<td>Connections</td>
<td>3.25</td>
<td>3.92</td>
<td>3.53</td>
</tr>
<tr>
<td>Students</td>
<td>16</td>
<td>13</td>
<td>17</td>
</tr>
</tbody>
</table>

Figure 4-8, Survey of Students Results

There appears to be an increase in all four averages from the first to the second lesson of the transformation inquiry. One can not draw the conclusion that these inquiry lessons significantly increased students’ curiosity, questioning, explaining, connecting of ideas. The two lessons were given to different groups of students (with a small N) so it is difficult to create a baseline for comparison. It is likely that with the second teaching of the lesson, the teacher’s questions and interactions with the students were more effective at shepherding students through the inquiry, as was seen in research about teaching IBL being its own professional development (Leikin, 2006). In all cases, it appears that the lesson did not take students to extremes, give that values of 3 or 4 are close to the “usual,” which we interpret to mean there is not likely to be much harm done by
switching to an inquiry mode of teaching. The survey results certainly do not indicate drastically lower rates of students asking questions, making connections, feeling curious, or explaining. Hence, these results are encouraging to teachers who wish to transition from textbook based learning to inquiry.
Chapter 5  Guidelines for Creating and Implementing IBL

This thesis has looked at the literature surrounding the history of inquiry-based learning and the growing body of evidence that IBL benefits students by reducing gaps between demographics and encouraging more students to take more mathematics. With this information in mind, a teacher may feel that switching from a traditional lecture-based method of teaching to IBL would benefit his or her students. This is no simple task for a teacher, and so this thesis will now present a guide for teachers to transition to inquiry-based learning. Such a transition may be accomplished with three approaches:

• adopting or modifying existing inquiries to insert into a geometry course
• transforming a textbook lesson into a guided inquiry worksheet
• creating an inquiry lesson from scratch

Note that the three approaches are listed in order of increasing depth of inquiry, and that order was chosen to ease the transition from one form of education to another. A teacher or a student may feel uncomfortable at the prospect of inquiry lessons due to the likelihood that “less” of the course may be covered. While a traditional march through curriculum via lecture might cover more topics, mathematics is not just a set of topics but also a set of practices. To truly think mathematically, students must find patterns and make use of structure. This leads a teacher to ask, “If structures and reasons are always given to students, will students be in a mindset to find structures within unfamiliar problems? Will students have the confidence to devise their own reasoning to support
their intuitions?” With inquiry, students may devote more time to and gain more experience in the mathematical practices, covering fewer topics but gaining useful practices they may apply to topics new to them in the future. This is what R.L. Moore’s student meant by “going from strength to strength” (Parker, 2005, p. 273). Moreover, this thesis looked at the literature surrounding IBL and found studies concluding that covering fewer topics had no ill effects on students (Laursen, 2013).

Because experienced teachers who shifted to IBL found that accommodations and patience were necessary to shift student appreciation for IBL over, this thesis proposes that aspects of inquiry may be gradually introduced with the aim of encouraging students to question and to conjecture mathematical ideas, to engage with mathematical ideas in ways beyond repetition of solution methods, progressing to deeper and deeper thinking. To this end of progressing toward deeper inquiries over time, the thesis will explore the three approaches on the list in order.

**Adopting or Modifying an Existing Inquiry**

It is key to remember that teachers who wish to begin with inquiry need not demand brilliant inspiration of themselves. Copying other instances of inquiry methods through the use of investigations and explorations provided by textbooks, other teachers, or professional organizations is acceptable, and a fertile place to begin. Within textbooks one may find “investigations” or “explorations” that offer a few pieces of information about an idea then a few questions that prompt students to discuss and draw conclusions. Geometry has some simple rules such as the total of the interior angles of an \( n \)-sided
polygon in degrees equals 180(n-2). The Bass Geometry textbook (2004) for high school approaches this question by providing a table of polygons with increasing $n$ (number of sides) to allow the students to think about a pattern in order to generate the formula. Another such question geared to this approach is to ask, “How many diagonals does a 100-sided polygon have?” The number of sides should be chosen to be so large that actually drawing it to count diagonals would be difficult, thus creating a cognitive need for the exploration of simpler problems with small numbers of sides to generalize a formula for all $n$. (From an early chapter of Michael Serra’s Discovering Geometry: An Inductive Approach) These are manageable activities for the teacher who wants to wade into the ocean, as opposed to diving in.

**On Choosing an Inquiry Lesson**

At this point, if a teacher is choosing to implement some inquiry-based learning, the nature of the choice must be considered. The teacher must judge the activity against certain criteria for deciding what makes good inquiry. First, *an inquiry should not state the idea that is to be discovered in an obvious way*. For example, the Bass Geometry book (2004) has an activity to let the students discover the formula for the sum of interior angles in a polygon—and on the next line of the same page is the formula that students are supposed to derive themselves (see Figure 5-1.) Such quick answers shut down student thinking and conversation. Teachers must watch out for this error in pedagogy.
Investigation: The Sum of Polygon Angle Measures

You can use triangles and the Triangle Angle-Sum Theorem to find the sum of the measures of the angles of a polygon. Record your data in a table like the one begun below.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of Sides</th>
<th>Number of Triangles Formed</th>
<th>Sum of the Interior Angle Measures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td></td>
<td>180</td>
</tr>
</tbody>
</table>

- Sketch polygons with 4, 5, 6, 7, and 8 sides.
- Divide each polygon into triangles by drawing all diagonals that are possible from one vertex.
- Multiply the number of triangles by 180 to find the sum of the measures of the angles of each polygon.

1. Look for patterns in the table. Describe any that you find.

2. **Inductive Reasoning** Write a rule for the sum of the measures of the angles of an \( n \)-gon.

By dividing a polygon with \( n \) sides into \( n - 2 \) triangles, you can show that the sum of the measures of the angles of any polygon is a multiple of 180.

**Theorem 3-9**  
**Polygon Angle-Sum Theorem**

The sum of the measures of the angles of an \( n \)-gon is \((n - 2) \times 180\).

Figure 5-1, An Inquiry with Immediate Answer (Bass, 2004, p 145)
Once this first simple criterion is met, a teacher ought to keep in mind more complex criteria when selecting or modifying an inquiry.

Our second criterion to assess a prospective inquiry is broader than the first: Do the questions of the activity encourage students to converse about the mathematics, to have their own conjectures, and to put their own ideas into words? Assuming the activity passes the “don’t give away the answer” criterion, the questions of it should poses enough challenge that students will need to talk, perhaps to wrap their minds around what the question is asking for, perhaps to suggest methods of solutions. Such questions that meet this criterion can be open-ended or even vague. For example, in the transformations exploration (see Figure 4-1) the opening question asks “What transformations happened here?” Many correct answers exist because any number of compositions of transformations could result in the pre-image and image on the paper, thus one suggested answer will not finish the conversation but begin it. One student might see a translation followed by reflection, but another might see exactly one reflection with a mirror line. Thus students will not only be confronted with the question on the paper, but also be confronted with others’ conceptions of the question, experiencing multiple perspectives of the same question. The vague or open-ended questions allow students to use their own mathematical experiences to react, rather than try to fit their thinking into a procedure handed down to them from an authority. And when the students are thinking about a problem on their own terms, that is inquiry.

Even if there are not open/vague questions, we may assess a prospective inquiry by a third criterion: Do the specific questions focus on a key idea? Because of the high
value of the vague or open-ended question, one might think that if the lesson has too many narrow questions, such as “What is the slope of this line?” or “What is the measure of vertex A?” then it may leave students trying to remember procedures instead of inquiring to get at the bigger picture. However, narrow detail-oriented questions can be desirable if they draw students’ attention to a general idea, like in the activity about the sum of the interior angles on an \( n \)-sided polygon. Another activity might ask a few narrow questions to lead students to recognize the slope of a line in its graph and then recognize the slope as a coefficient of \( x \) in the linear equation of the form \( y=mx+b \). After seeing a few of these narrow questions, a teacher should expect to see followup questions that ask students to generalize from examples and to apply this new generalization. Or, consider how the Exterior Angle exploration (see Figure 4-1) starts with the same question about three specific triangles, then asks students to recognize a general rule for finding an exterior angle in any triangle. The structure of a few narrow, detail-oriented questions leading to a deeper more general idea is a great pattern for inquiry because whenever mathematicians see one or two things are true, they wonder if these truths hold in general as well. Teachers can use this pattern as a template for transforming a lesson from a textbook into a structured inquiry worksheet.

The image of the wondering mathematician leads us to a fourth principle: *Does the inquiry teach or prompt students to ask more questions?* The goal of a math lesson ought to be to get students digging down into the lesson topic, inquiring as to what exactly is true here. The goal of a math course ought to be more than the sum of its lessons, to give students mental tools with utility to transfer beyond the problems they
have seen in the course. A teacher ought to look for questions which ought to be asked in any general attempt to solve a problem:

- What value/ratio/rate do you seek?
- What information have you been given?
- What equations or formulas do you think might apply or might be useful?
- What bounds must exist on the answer?
- What would be an answer that is too high?
- What would be an answer that is too low?
- Would the answer be positive or negative?
- What details could tell you that you are closing in on a solution?
- Now that you have a result, how can you confirm it?

After experiencing one or two inquiry lessons with these sorts of questions, a teacher would be well-advised to point out to students the wide applicability of such questions and the power they give the student to become more certain in solving problems. Thus the previous list is a set of questions that students would see and adopt (and adapt) for themselves. Essentially, one of the teacher’s goals is for students to internalize questioning, both by copying the problem-solving questions above as well as creating their own questions. To do this, look for an inquiry to prompt students to create their own questions because this will develop even more autonomy within students. When examining an inquiry, look for prompts like “What questions do you now have?”
or “What would you like to know about…?” Certainly these fall into the category of open-ended and vague questions, but these particular vague questions are developing student autonomy in making sense of mathematics.

More sense-making questions to look for within an inquiry might be along the lines of confirmation and variation:

- Does this work if we change the angles by a few degrees (length by a few millimeters)? Suppose this one aspect changes, what then?

- Yet other questions might move beyond small variations to larger consequences.

- If this statement is true, then what else is implied?

- What assumptions does this statement rest on?

- What can you generalize from this?

Such questions have tremendous value in building a student’s mathematical mind because before conjectures can be proved, one must really probe aspects of the conjectures to determine where the boundaries of the conjecture are, what its structure is and what assumptions give it validity.

A fifth desirable characteristic in an inquiry is, *Do the questions encourage students to connect ideas?* The Exterior Angle exploration, on its second page, leads to students to use this theorem to get the sum of the exterior angles of any triangle. This is a relatively small step to connect the single angle with the sum of all the angles. But inquiry can help students make connections across items in different categories. This gets
at the fundamental nature and power of mathematical theorems. Consider how the Pythagorean theorem starts with a measurement in the category of angles and ends up with measurements in the completely different category of lengths. “If a triangle has a right angle, then the square of the length of the longest side is the sum of the squares of the lengths of the two shorter legs.” Or consider this: If parallel lines are crossed by a transversal then corresponding angles are congruent. If a quadrilateral has two pairs of parallel sides, then its diagonals bisect each other. This last theorem connects direction (parallel sides have the same direction on the boundary of the polygon) with equal lengths (the two congruent halves of each diagonal on the interior.) This principle applies beyond geometry: The Fundamental Theorem of Calculus connects slopes to areas and (in a broader sense that extends into multivariable calculus) connects boundary information to information inside a region. The power of mathematical thinking is about so much more than mere calculations. Inquiry allows students to experience how mathematics gives us truths that are not obvious. This principle of “surprising connection” can be seen in the inquiries involving perpendicular bisectors to find the center of a rotation and to determine if a polygon is cyclic.

To summarize, the choice of an inquiry means a teacher needs to look for good questions that probe a variety of levels of thinking, from simpler levels like judging the reasonableness of an answer to wondering about connections with ideas outside the lesson. Five criteria can help a teacher select a good inquiry:
• Does the inquiry give away the big idea, shutting down conversation?

• Does it have open questions that all multiply ways for a mind to engage?

• Do the specific question focus on or lead to a key idea?

• Does the inquiry encourage students to generate their own questions?

• Does the inquiry lead to connections between ideas?

If an activity meets some or all of these criteria, the teacher can expect it would encourage students to think and discuss their ideas, fostering inquiry in the class.

Once a teacher has chosen to implement such a lesson, a teacher should prepare by attempting to do the textbook investigation themselves step by step (or perhaps with a partner to simulate the student conversation one hopes for) as preparation for the class activity, noting their own moments of confusion or questions. What sort of errors might students make? Let us focus on a lesson (see figure 5-1) to arrive at the concept that the Interior Angle Sum of a Polygon is \((n-2)180\). The lesson asks students to draw non-overlapping triangles inside the polygons. A teacher could anticipate that some students will draw triangles that do overlap, thus driving the process off track. Overlapping the triangles is essentially counting some angles more than once. After anticipating errors, teacher may create hints to help students progress past their own moments of confusion.

For this activity, the students should understand the parts of the formula.

Teacher prompts, “Where does the 180 come from?”

Hint: What shapes are we breaking the polygon into? Answer: Triangles.
Hint: What’s the sum of 3 interior angles of a triangle? Answer: 180 degrees (oh!)

Teacher prompts, “Where does the (n-2) come from?”

Hints: point to a pentagon, say “When there are 5 sides, we break it into …” “3 triangles”. Then point to a quadrilateral, say “When there are 4 sides, we break it into…” “2 triangles.” Then ask, “When there are n sides, we break it into …” “n minus 2 triangles.”

Finally, bring the focus back to the big, tough question you led with: “What’s the sum of all the interior angles of a 100-gon?” Expect students to answer with a complete sentence because a complete sentence is a full thought, not merely a juxtaposition of fragments.

When students are challenged to observe a pattern and use it to predict (instead of merely recalling a procedure) then they gain the experience of seeing how a set of facts can lead to broader ideas. The inquiry into the nature of a formula can demystify the function, essentially showing the students the gear mechanism inside the black box.

If a teacher copies this lesson straight out of the book, he or she may want to add questions. Certainly, summative questions that require evaluating the formula for any given number of sides, like 17 or 25 could be done, and students could be given a total such as 1800 degrees and be asked to find the number of sides. Another creative question could ask “Does a polygon exist with a total of 2000 degrees in its interior angles?” When students solve the equation they will get a non-integer number of sides. What does that mean? If a student is thinking, she or he might say “no, polygons must have an
integral number of sides.” The role of a teacher in inquiry is not merely to teach rules, but to get students think about breaking mathematical rules. This, too, is a mathematical practice in the tradition of Saccheri and the other thinkers who opened up non-Euclidean geometry by inquiring into Euclid’s Fifth Postulate (aka the Parallel Postulate). It is anticipated by the author that when a teacher selects and prepares inquiry lessons from others, the teacher may feel the desire to create their own questions as modifications, so we now look at the second of the three approaches, which requires a little more creativity from a teacher.

Transforming a Textbook Lesson into Structured Inquiry

As a further step beyond the use of an already published exploration, a teacher may design a structured inquiry worksheet (also referred to as structured discovery worksheet) by transforming existing lessons from textbooks or lecture notes. After choosing the goal of the lesson, the teacher may create simple questions that lead toward the idea, then ask the students to look over all the exercises and see a pattern or relationship. For instance, after students have learned that a linear pair of angles is supplementary and the triangle sum theorem (the sum of the interior angles of a triangle is 180 degrees), they may be given a guided discovery worksheet (see Figure 4-1) to lead them to the Exterior Angle Theorem: The measure of an exterior angle equals the sum of the two remote interior angles of the triangle. The worksheet might start with 2 or 3 specific triangle diagrams with 2 given interior angles and 1 unknown exterior angle to be found. It is anticipated that students will find the exterior angle in 2 or more steps via the
triangle sum theorem, and then later a question may be written on the sheet, “Can you find a more efficient way to get the exterior angle from the given angles?” If a teacher is feeling more brave, they may phrase the question more generally, “Do you notice a relationship between the angles?” Or the worksheet may end with a summative question, using only variables for the two interior angles and asking for an equation to be written to generalize this theorem. The key to creating such a guided discovery worksheet is understanding how the more advanced (final) idea may be achieved through steps involving more basic ideas, and providing enough examples that students may develop the advanced idea out of the basic examples. Then, importantly, summative or general questions must be asked so the students no longer require the steps involving the basic ideas, instead the advanced idea becomes the primary mental tool. It is the task of generalization or summarization that gives inquiry the advantage over mere drill because the students have their awareness drawn to an abstraction of commonalities among examples. Because the task of writing an exploration may seem daunting, the teacher should take longer view of the writing process, accepting that initial forays into writing inquiries will need revisions which can be implemented in a later term. After the students do the worksheet, ask students for their opinions on how to make the wording of the questions clearer, then rewrite the worksheet for the next term. Note how, in the past paragraph, the questions that were suggested for an inquiry met the question criteria for choosing a good inquiry. A teacher who moves from choosing inquiries to writing guided inquiries will still use the same criteria, but in a more creative way.
This second approach was entitled “transforming textbook lessons” because it assumed that early forms of creativity to some extent rely on copying and derivation. Teachers will likely have textbooks available with lessons they may read and then rework into structured inquiry worksheets. We now examine the advantages and opportunities that inquiry-based learning offers over textbooks by comparing two possible lessons with the same goal. The first based on the Prentice Hall Geometry textbook, by Bass (2004,) and the second a guided inquiry worksheet (Exterior Angle Theorem, Figure 4-1.) See Figure 5-2 for the textbook approach.
Parallel Lines and the Triangle Angle-Sum Theorem

OBJECTIVE
2

Using Exterior Angles of Triangles

An exterior angle of a polygon is an angle formed by a side and an extension of an adjacent side. For each exterior angle of a triangle, the two nonadjacent interior angles are its remote interior angles.

The diagram at the right suggests a relationship between an exterior angle and its two remote interior angles. Theorem 3-8 states this relationship. You will prove this theorem in Exercise 49.

Theorem 3-8
Triangle Exterior Angle Theorem

The measure of each exterior angle of a triangle equals the sum of the measures of its two remote interior angles.

\[ m \angle 1 = m \angle 2 + m \angle 3 \]

Example
Using the Exterior Angle Theorem

Algebra
Find each missing angle measure.

a. \[ \begin{align*}
40^\circ + 30^\circ &= m \angle 1 \\
70^\circ &= m \angle 1
\end{align*} \]

b. \[ \begin{align*}
113^\circ &= 70^\circ + m \angle 2 \\
43^\circ &= m \angle 2
\end{align*} \]

Figure 5-2, Textbook Examples (Bass, 2004, p 133)
In Figure 5-2, we see the general idea presented with clear definitions of “exterior angle” and “remote interior angles,” plus a clear visual explanation of the theorem. The textbook offers two examples with specific degree measures. Because the relevant exercises will have citations of relevant examples next to them, students will not have to struggle to find the template for their solutions. This is mimicry made easy.

However, the structured inquiry worksheet (Figure 4-1) begins with three specific examples of triangles and their exterior angle. Because it is possible to find the unknown exterior angle without knowing the Remote Exterior Angle theorem, students have the opportunity to look for and express regularity in repeated reasoning and to look for and make use of structure (CCSS 2010), to take the initiative and start thinking right away. Working with partners allows the students who forget background knowledge (such as the sum of the interior angles of any triangle or the idea that linear pairs of angles sum to 180 degrees) the chance to communicate with others and get themselves up to speed with their peers, finding the exterior angles.

Note how the structured inquiry worksheet allows the student to generalize because generalizing occurs after 3 concrete examples have been worked through. The students’ minds are the ones making sense of the angles and looking for the structure of the shape, whereas in the textbook the authors give the structure to students (you can not really look for something if someone else hands it to you.) In the structured inquiry worksheet, the statement of the theorem happens at the end of the experience of finding exterior angles, so the students’ minds are drawing a common rule out of a few examples. Because the worksheet asks for it, the students must summarize their thinking
themselves. Their minds have a chance to do the higher-level thinking that is summarization. In conclusion, a structured inquiry worksheet can lead students to an “a-ha” moment with deeper understandings beyond mere memory of recipes. It is more likely than a textbook to engage students in the mathematical practices such “look for and make sense of structure (CCSSI 2010.)” However, teachers may want to consider lessons that are even more open, that give students even more autonomy in their learning. Beyond the pre-written worksheet of the structured inquiry is a lesson that requires more creativity of the teacher—a guided inquiry.

Creating a Guided Inquiry from Scratch

This is the third approach to adding inquiry to a geometry course, and this approach requires more creativity than adopting an IBL lesson or writing a structured inquiry worksheet. Two examples of this approach are provided in Chapter 3 of this thesis, entitled Transformations and Cyclic Polygons. With only a minimal number of preprinted questions on the paper in front of them, students are freer to apply their own perceptions, questions, and thoughts to the shapes before them and the teacher will need to be mindful in guiding the class to be sure students arrive at the desired mathematical ideas. To write such lessons, teachers may spark their own creativity by inquiring into math themselves.

Combinations of unlike ideas are good for inquiry. Typically the constructions unit of Geometry is taught separately from the transformations unit in a course. But there is a way to use perpendicular bisectors to locate the center of a rotation or to locate the
mirror line in a reflection, which students might discover in the inquiry lesson on
Transformations in Chapter 4. How did the Transformations Inquiry (see Figures 4-2 and
4-3) come about? The author had seen some Geometry teachers from a different school
display expert construction techniques in a professional development meeting. Also, it
was disappointing how infrequently the tools of compass and straightedge were used
outside of the one Constructions chapter of a Geometry textbook, and the author
wondered if the tools could be applied to later chapters of the book, so he began to play
with a compass and a paper that already had a reflection printed on it. Very quickly, he
realized how a perpendicular bisector could find the mirror line, so then the next question
to come to mind was, Can rotation be understood with compass and straightedge? By
inquiry, a new lesson was created. A teacher interested in more IBL might invent similar
tasks for translations or dilations, involving compasses and unmarked straightedges,
which we leave as an exercise to the reader. This thesis suggests the teacher get some
paper and play. Other combinations of topics will now be suggested.

Another possible connection for topics within geometry may be with the
connection of constructions to postulates and theorems. This is hardly a new suggestion,
as any reading of Euclid’s First Book will show that Euclid executed constructions in
conjunction with triangle congruence postulates (side-angle-side, side-side-side, CPCTC,
etc.) Many teachers already do this (as do many textbooks,) but in some curricula or
some classrooms these ideas get separated into silos so students do not think about the
connections between them. After students learn how to construct a copy of an angle (with
compass and straightedge), then they may be asked to prove it (using SSS and CPCTC).
SSS is short for the side-side-side congruent triangles postulate and CPCTC is an abbreviation for “corresponding parts of congruent triangles are congruent.” In this way, the constructions of Geometry are linked to the proofs of Geometry. Teachers who wish to create new inquiry lessons should consider the chapter titles of their Geometry course and ask themselves, Which two chapters seem unrelated? How can I connect them?

And just as much as combining disparate ideas is fertile ground, so to is removing a piece of an idea. The transformations are often taught with coordinate geometry and the algebra rules in terms of points (x,y). But a teacher could set up transformations omitting the coordinates. In my Transformations lesson on reflection and rotations, the coordinate grids were removed, creating a need for the compass and straightedge. Here is an undeveloped idea that teachers might find worth of inquiry: Traditionally, students are taught how to construct medians of triangles and the medians’ intersection point is the centroid (center of mass if the triangle were a uniformly thin piece of metal). What if this middle step of constructing medians is removed? The lesson asks, “Given a triangle, and the point inside it which is its center of mass, how do you find the center of mass of other triangles?” Will students come up with constructing the median as their method? Or will they assign a coordinate system and find another method? Inquiring minds want to know.

Another way to think about removing a piece of an idea is burying the main idea and parceling out just enough information so that students can uncover it. Instead of telling geometry students that perpendicular lines have opposite-signed reciprocal slopes, give students a set of three graphs each with a pair of perpendicular line segments on them, then ask them to find find the slopes of the lines and ask, “What do you notice?”
This method of removing information or loosening constraints might be an easier way to ignite creativity with lesson plans than combining unlike ideas. There may not be a rich, easily teachable connection between any two unlike topics, but there are many richly connected topics and questions already in geometry curriculum which can have part of them “removed” or “covered up,” so students might be delighted to search for these ideas.

A third way to stimulate inspiration would be for teachers to “open the question.” This phrase means teachers should read examples or lessons in a book with an eye to notice (and then remove) the constraints. For example, in the Discovering Geometry book (Serra, 1997, p 129) there is simple task asking students to draw 2 quadrilaterals intersecting in 1,2,3,4,5,6,7,8 points. But if the question were worded more openly, “Two quadrilaterals can intersect in 1 point or 2 points or maybe more. How many intersection points can you make?” This more opening wording of the task allows students to make conjectures, to check each others’ work, to demonstrate and to critique, as students do in the four methods examined in Chapter 2. Perhaps some conjectures will be wrong, such as “it is impossible for 2 quadrilaterals to have exactly 3 intersections,” but it is valuable mathematical practice to winnow out false conjectures in the quest for provable truths. Moreover, by opening the question, the teacher may give students the chance to take the idea further than the textbook author anticipated, breaking an assumed constraint to find more than 8 intersection points between two quadrilaterals. A common assumption when quadrilaterals (or other polygons) are discussed is that the quadrilateral is convex. And
with convex quadrilaterals, the maximum number of intersections is 8. But two concave quadrilaterals have the possibility of intersecting at 16 points.

Varying the parameters or loosening constraints can happen in other ways. A teacher could prompt, “Create triangles such that the centroid is (3,5) in the coordinate plane.” Do you notice something about the coordinates? Because the centroid is the average of the coordinates of the 3 vertices, students might hypothesize that polygons beyond triangles can have centroids. “Create a quadrilateral or a pentagon with a centroid at (3,5).” Then the teacher may ask students to justify their answers, engaging the students in a mathematical practice of recognizing and making use of structure as they find out how coordinates of vertices relate to the centroid.

The Cyclic Polygons or Not? Lesson (See Figure 4-4 through 4-7) illustrates an inquiry lesson created by a combination of loosening constraints within another lesson. The author observed a lesson online about perpendicular bisectors and concurrency by Sam J. Shah (2014) requiring the use of Geogebra with a structured inquiry worksheet. Shah’s lesson sets up students to learn first that it is rare for a polygon’s sides to have concurrent perpendicular bisectors, then to be surprised that triangles’ sides always have concurrent perpendicular bisectors. By contrast, the author of this thesis loosened the constraint of needing Geogebra and computers for this by turning it into a paper and pencil and compass and ruler task and lowered the entry point by switching the focus away from perpendicular bisectors (a somewhat advanced mathematical tool) to circles, which are a very simple, intuitive geometric concept. By shifting the focus to circumscribing polygons, the task becomes more accessible, especially to students who
take the initiative to use visual estimation to guess a circumcenter and its circle (a few students guessed successfully when the lesson was taught.) Also, the Cyclic Polygons or Not? lesson allows student to think of the perpendicular bisector themselves, as a tool to find the center. A few paragraphs later in this thesis we will see how the Cyclic Polygons or Not? lesson uses another mathematical process of reversing processes. As mathematical problems often require students to engage in several mathematical practices (as described by the Common Core) at once, an inquiry lesson created by a teacher may meet more than one of the criteria described in this thesis.

Opening the question may be taken to an extreme. A teacher may decide to ask, “I will give you the answer. You give me the question.” But, instead of one question being right, multiple student questions can be put forward for a gallery walk. “The area is 450 square units.” What questions will students ask? Will students create a simple rectangle with 9 units and 50 unit dimensions? Will students create a circle with that? Will students create an annulus? Will they create the surface of a sphere? Will students combine polygons? Some of these questions demand little cognitive effort (the rectangle) but others require more, and the task opens up possibilities for students to challenge themselves and be creative in a way that many textbook questions never do. This requires the teacher to listen (or read or observe) to the students’ explanations carefully. The gallery walk will require students to observe each other’s ideas, and provides a chance to critique, to have an opinion about solutions much like people have opinions about art. It is demanding of the teacher, who must ask “in looking at this question, what evidence is there that the student has mastered the concept of area? Does the student’s understanding
of area include symmetry and dissection? Does the student appreciate how area is conserved by shear transformations?” Inquiry-based learning does not just mean the student inquires—it also means the teacher must inquire into student reasoning and questioning.

Another guideline for inquiry has to do with reversing the order of processes, or essentially considering inverse functions. The lesson on cyclic polygons begins with the rather simple procedure of laying points on a given circle to create a polygon, which is a trivial process. But the inverse of that is non-trivial and can be worthy of inquiry. Given points that are vertices of a polygon, can a circle be drawn through all the points? It takes time and experimental trials to start to know how to answer that question. And inverse processes are a common form of thinking throughout mathematics. It is trivial to multiply numbers but non-trivial to factor a given number. It is trivial to input values for x into a polynomial and find output values for y, but not so trivial to start with an output value for y and deduce one or more values for x. It can be simple to execute a transformation by inputting coordinates into a rule, but more challenging to deduce a rule from the given coordinates of the pre-image and the image.

It is hoped that these guidelines for creating inquiry (combining unlike ideas, removing a piece of an idea, opening the question, and investigating the inverse process) will help teachers to ignite their own interest and their students’ interest in geometry in the secondary school years and later on.
Still one question ought to be addressed if a teacher is going to institute inquiry lessons: What is going on in the teacher’s mind during an inquiry lesson?

**Best Practices for Teacher Habits of Mind during Inquiry**

What is going on in a teacher’s head while teaching inquiry? The current research has illuminated this question, but before reviewing the research, it will be illuminating to consider a pattern established earlier. We saw that for the teacher seeking to move their teaching style from traditional lecture to IBL, the possible styles of lesson-planning ramped up from adopting existing inquiries to transforming traditional textbook lessons into structured inquiry to outright creation of guided inquiry, rising in order of increasing freedom for students’ minds. This “ramp” allows teachers to maintain their comfort as they venture forward into an uncertain new style, giving them time to examine and reflect on the changes they are instituting. Like the ramp of planning lessons, we advise a ramp of teacher thinking during inquiry. We will examine first guidelines for a beginning teacher, discuss how to build on more guidelines as a teacher grows comfortable, then conclude with a set of practices from the literature, and a case study from the literature which affirms that it is possible for teachers to grow proficient at IBL, using their own teaching as if it were professional development.

After the lesson has been written and photocopies have been passed out, and the question has been presented to the students, what should the teacher keep in mind as the students begin their mathematical inquiry? The first guideline to is encourage student talk about math. This means minimizing teacher talk to allow time and space for student talk,
and making sure that there is a low bar to jump over for all students to get their minds into the problem. For instance, in the Cyclic Polygon or Not? lesson, every high school student knows what a circle is, what a polygon is, and they can conceive of a polygon’s corners touching the circle, even if some students would have trouble writing a definition of such vocabulary terms. The teacher needs to put the prompt out before the class and let the students begin to wrestle with the problem. Perhaps some students will have difficulty understanding the prompt, so teacher might identify such students and invite student talk by asking quietly, “Would you like to ask a clarifying question?” Most importantly, if a teacher wants students to talk, then the teacher should listen so the students know their speech is valued. As the next few paragraphs describe more and more practices for a teacher to manage an inquiry, the complexity and number of the practices may seem daunting, but no matter what, the teacher should be listening to what the students say, all the while keeping the goal of the lesson in mind without giving it away (as we saw in our criticism of a textbook earlier.)

Once the teacher believes the students are productively engaged, the opening game is over and the middle game has begun. This will consist of student explorations of the concepts and misconceptions related to the inquiry. Essentially, mistakes or misconceptions are not pitfalls to be jumped over. Mathematicians are not commandos rushing through a jungle to hit their target then be extracted by helicopter. They are engineers exploring the land so they can build a road or bridge on it. The sturdier the road, the more heavy traffic it can support in the future. Misconceptions and valid concepts are like natural formations in the land, and an engineer wants to know if the
earth underfoot is solid enough to support a road or bridge piling. Maybe this patch of
ground is too soft and any pavement laid on it will sink, crack, or be washed away. So
students need to probe down to know where the solid ideas are, where the foundations
can be laid for a lasting, trustworthy path. To put these metaphors into plainer language:
misconceptions are opportunities for students to discuss what they understand, not
something to be avoided. When they explore a misconception, that firms up their
knowledge of math to make it more lasting, more certain, and more useful.

Thus, a teacher in an inquiry class needs to anticipate misconceptions and write
questions that could be used to help students see their errors and validate their
conjectures. It would burden a teacher to try to keep a list of one dozen misconceptions in
mind. So, only one or two misconceptions are worth keeping in mind, but the teacher’s
mind ought to be alert for other things. A teacher is going to have to parse student
conjectures to recognize key similarities to the lesson objective, so a better guideline for a
teacher in the midst of an inquiry lesson is to maintain a default attitude that what the
student is saying is valid and logical. If statements or work by students do not seem on
their face to be correct, wait before saying “No.” Ask for more explanation or a
restatement of the conjecture or reasoning. Two advantages result from this strategy of
hesitation to correct. First, given a little more time, students may achieve their own
correction, which is much more valuable because it is a realization from their own
thought, and that experience enforces the autonomy of the student. Secondly, students
may actually be correct, but the teacher will not realize it because differences in
background may muddle the communication of the idea (and one must assume that a high
school teacher who has completed college-level mathematics courses has a different background and diction than a high school student). Given a little more time, and a teacher’s non-judgmental prompt like, “would say your idea in a different way?” or “what I heard you say is _____, is that what you meant?” the student may clarify without feeling negative emotions like shame that can shut down focus on the math. Obviously, the teacher’s knowledge of the topic will be in a more formal wording than the students’ questions and conjectures will, so the hermeneutic of believing student conjectures are right is important to help the teacher look and listen for evidence that the students are getting close to the lesson goal.

This act of monitoring student work might be helped by the teacher taking notes on a clipboard to help the teacher track which students had which insights, and which ideas could be stressed (and in what order) as the lesson concludes. An inquiry demands higher cognition from a teacher than traditional lessons, in which a teacher might need only to keep an answer key handy for the fifteen exercises students are supposed to do in class. By contrast in the inquiry, the teacher must keep in mind the goal of the lesson, track the progress of individual students or student groups, make judgments about when to intervene by offering hints or questions to help students’ minds make the right connections, and finally judge the right moment to call the class together for comparing solutions and reflecting on ideas. As a final piece of advice, a teacher may look at the clock, and decide to intervene, offering clues or questions to guide students to the goal of the lesson before the class period ends. Just as IBL offers students the freedom to try new ideas for themselves and risk making wrong decisions, so too must a teacher of IBL risk
making wrong decisions, either by intervening too soon or not at all in students’ inquiry and discussions.

The goal of the habits of mind of a teacher in the midst of IBL is for lessons to have a thorough plan of action for considering students’ solutions and allowing students’ to consider each other’s solutions. And within the literature of IBL, an excellent set of useful practices have been offered by Stein, Engle, Smith and Hughes.

Specifically, the five practices are: (1) anticipating likely student responses to cognitively demanding mathematical tasks, (2) monitoring students' responses to the tasks during the explore phase, (3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase, (4) purposefully sequencing the student responses that will be displayed, and (5) helping the class make mathematical connections between different students' responses and between students' responses and the key ideas. (Stein, 2008)

The Stein article offers examples with a case study of an elementary math classroom, and the reader is encouraged to study it carefully. These five practices would seem daunting to a teacher new to IBL, and so it would best for the IBL teacher to begin slowly and not demand perfect execution of the five practices in the early iterations of teaching with IBL, just as one would not expect a new IBL teacher to devise a creative new inquiry early on.

We have reviewed the list of increasingly sophisticated ways to introduce IBL lessons, and the list of increasing complexity in how to mindfully monitor an inquiry in the midst of class. As mentioned before, the benefits of IBL seem attractive, but the challenges of implementing it may seem daunting. Research suggests that collaboration
by a group of teachers using experimentation and reflection in lesson planning can be effective professional development (Witterholt, 2012) so we can expect that this could be one path to more IBL. And if collaboration proves difficult, one case study of an elementary teacher introducing IBL offers inspiration. The teacher gradually improved in managing inquiry-based lessons and the students’ ability to cope with the inquiry tasks improved as the teacher’s proficiency in managing inquiry-based lessons developed. Why? First, as the teacher “became aware of the students’ progress she grew more confident about granting them independence. Second, [her] growing expertise in managing inquiry-based lessons accelerated students progress and active learning.” (Leikin, 2012) Over a period of sixteen months, a teacher’s “proficiency developed in the course of her teaching experience without any professional development intervention.” (Leikin, 2012) In essence, the authors of the case study indicated that “the act of teaching presents an opportunity for teachers to learn.” We think that it is possible for teachers to implement IBL successfully (either independently or with a cohort of teachers), exploring IBL methods and improving over time. At the present, it seems as though the decision-making about when to intervene is more of an art, which a teacher can improve over time, and so it remains for researchers to develop models of decision-making about when teachers intervene during inquiry. It seems especially challenging to develop decision-making models that could be applied across many lessons. While this topic remains to be researched, Leikin’s work gives inspiration that a teacher may begin a successful transition to IBL without waiting.
Chapter 6  Summary and Conclusions

In this thesis, we have seen critiques of traditional direct instruction of mathematics, which relies on lectures and textbook examples, as training students to memorize and mimic procedures, and we have noted how inquiry-based learning (IBL) offers an experience in which students “do mathematics” in more creative and thoughtful ways. Can secondary school teachers, after seeing the advantages of IBL, transition their practices away from textbooks toward more inquiry? To answer this question, the methods, challenges, and effects of IBL were considered.

In Chapter 2, we saw that the Moore Method and Modified Moore Method utilize classroom time for teachers to pose mathematical questions and for students to present and critique their own proofs of theorems and solutions to problems. We reviewed how Harkness teaching has students’ questions, discussions, and solutions at the center of instruction, and how Harkness teaching requires that a teacher never give struggling students the next step but prompt them with questions that lead to their own realization of the answer. We considered the Interactive Mathematics Project, which promotes teamwork, discussion, strategic choices, and independent thinking. All four of these inquiry methods have the purpose of fostering creativity and critical thinking in students, developing what R.L. Moore called “power,” a capacity to think and solve problems that have not been explicitly taught. While IBL courses may cover fewer topics or deliver less content, the advantages of gaining “power” are that students gain a deeper understanding
of and appreciation for mathematics. This belief has been supported by a study of college students in IBL which showed no negative effects by a course covering less material (Kogan, 2014). Furthermore, a growing body of research shows IBL boosts student interest in mathematics, especially among women, as well as reduces achievements gaps between low- and high-achieving students, as compared to traditional lecture methods (Laursen, 2013, 2014). We found that the preponderance of evidence suggests teachers should take steps to incorporate inquiry into mathematics instruction.

In Chapter 3, we explored the barriers to entry as teachers shift into IBL. From interviews with working and retired teachers, we saw that students, administrators, parents, and guidance counselors can exhibit resistance to IBL. Advisable approaches to dealing with the barriers involved teachers listening for signs of concerns from others, communicating the benefits of IBL, and preparing changes to assessments and procedures. A transition is not simply a matter of substituting a new set of math problems for the old direct instruction curriculum.

In Chapter 4, three example lessons are offered, illustrating the styles of structured inquiry and guided inquiry. The structured inquiry took the form of worksheet in which students are prompted to solve simple exercises first, then to notice and to generalize a theorem about the remote exterior angle of a triangle through a structure of leading questions on the worksheet. Two lessons in the guided inquiry style encourage students to use construction tools to locate features of transformations such as mirror lines and centers of rotation. The guided inquiries begin with one question on the paper, and then the teacher may use a script to help create more of a conversation among
students about the mathematical ideas behind the problems, offering more freedom to students while still guiding them toward the goal of the lesson. These lessons were taught to high school students in direct instruction geometry courses, and student surveys were conducted to measure if students felt more curious, asked more questions, explained their thinking more, or understood more the connections between ideas. Survey results do not indicate drastically lower rates of students asking questions, making connections, feeling curious, or explaining. Such results are encouraging to teachers who wish to transition from textbook based learning to inquiry.

Because a transition to IBL will require more than three lessons, Chapter 5 presents criteria for teachers to choose suitable classroom resources as well as guidance for the creation of inquiry-oriented tasks by transforming traditional textbook lessons into structured inquiry worksheets or by creating guided inquiry lessons from scratch. For selecting an inquiry, teachers are advised to avoid inquiry that gives away the insight so quickly students need not think very long and to select for inquiry that focuses on a key idea, asks open-ended questions, connects ideas, or inspires students to generate their own questions. Beyond selecting inquiry lessons, the thesis explores how traditional textbooks may become a resource for structured inquiry by inverting the usual order of explanation followed by examples. A teacher may first ask students to try to solve two or three examples, then make conjectures about the idea underlying the examples, test their conjectures and prove them. Thus, the teacher has used the examples from the textbook to create opportunities to create and think on deeper levels than if students merely read the text. Teachers may also adapt guided inquiries from other courses, or they may create a
guided inquiry from scratch. Such creativity can be sparked by teachers seeking to
combine unlike ideas, by loosening constraints on questions, or by offering connected
ideas with the connection hidden for students to uncover. Finally, Chapter 5 offers
guidance on habits of mind for a teacher while teaching inquiry lessons to assure students
are making conjectures and considering each other’s ideas. While the task of transitioning
to inquiry may seem daunting, research has shown that teachers can improve their
proficiency at managing IBL (Leikin, 2006). We conclude that the transition to IBL can
succeed despite certain barriers, and provide students more experience “doing
mathematics.” Therefore, more teachers should make the transition, independently or
with colleagues.

As IBL practices spread, further study is needed to see if the gains measured in
student interest at the undergraduate level can be reflected in secondary education, and
further research could determine the efficacy of IBL in terms of helping low-achieving
students close gaps with high-achieving students. Research remains to be done to
understand better decision making about when teachers should intervene as students
struggle during inquiry lessons. Perhaps the most intriguing thought for further study is
the question: with more inquiry in mathematics courses at the high school level, how
should standardization of curriculum and testing accommodate this form of education and
the advantages it offers?
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